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Journal of Pure and Applied Algebra 191 (2004) 157–180
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

# Minimal filtered free resolutions for analytic $D$-modules

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Received 1 September 2003; received in revised form 25 November 2003
Communicated by T. Hibi

**Abstract**

We define the notion of a minimal filtered free resolution for a filtered module over the ring $\mathscr{D}^{(h)}$, a homogenization of the ring $\mathscr{D}$ of analytic differential operators. This provides us with analytic invariants attached to a (bi)filtered $\mathscr{D}$-module. We also give an effective argument using a generalization of the division theorem in $\mathscr{D}^{(h)}$ due to Assi et al. (J. Pure. Appl. Algebra 164 (2001) 3–21), by which we obtain an upper bound for the length of minimal filtered resolutions.


*MSC:* 16S32; 32C38; 16E05

In this paper, we study finite type left modules over the ring $\mathscr{D}$ of germs of analytic differential operators at the origin of $\mathbb{C}^n$. Such modules $M$ admit a finite length free resolution

$$\cdots \to \mathscr{D}^{r_2} \to \mathscr{D}^{r_1} \to \mathscr{D}^{r_0} \to M \to 0.$$

Free resolutions are a basic tool for studying $\mathscr{D}$-modules. For example, in order to compute the restriction or the local cohomology of a $\mathscr{D}$-module, we need a free resolution adapted to the V-filtration (see e.g. [6]).

However, such resolutions are not unique and it is difficult to define the notion of minimal (filtered) resolution of a $\mathscr{D}$-module directly. For this purpose, we first introduce

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0022-4049/\$ - see front matter 
doi:10.1016/j.jpaa.2003.12.005

in Section 1 the homogenized ring of differential operators $\mathscr{D}^{(h)}$ after [1]. This is a *graded* ring, which is, in fact, nothing else but the Rees ring for the filtration of $\mathscr{D}$ by the usual order of operators. We define the notion of a graded minimal free resolution of a graded $\mathscr{D}^{(h)}$-module in Section 1.

In Section 2, we consider filtered graded $\mathscr{D}^{(h)}$-modules with respect to a general weight vector $(u,v)\in\mathbb{Z}^{2n}$ for the variables $x$ and the corresponding derivations $\partial$. Then we define the notion of a $(u,v)$-filtered graded minimal free resolution for such a module. We prove the existence and uniqueness of minimal resolutions by using in both (non-filtered and filtered) cases a graded (resp. a bigraded) version of Nakayama's lemma in a way inspired by Eisenbud [3]. In particular, a minimal V-filtered resolution of the local cohomology group supported by $t=f(x)$ provides us with a set of numerical invariants attached to the divisor defined by a germ $f$ of analytic function (see Examples 2.11 and 2.12).

In Section 3, we show how to come back from $\mathscr{D}^{(h)}$-modules to $\mathscr{D}$-modules. The dehomogenization gives an equivalence between the category of graded (resp. $(u,v)$-filtered graded) $\mathscr{D}^{(h)}$-modules having no $h$-torsion and the category of $\mathscr{D}$-modules equipped with a good filtration with respect to the usual order (resp. a good bifiltration with respect to the usual order and the $(u,v)$-order). Therefore, we obtain, by dehomogenizing the previously obtained resolution, a finite free (resp. a $(u,v)$-filtered free) resolution which is canonically attached to any $\mathscr{D}$-module equipped with such a filtration (resp. a bifiltration).

In Section 4, we prove a generalization to submodules of $(\mathscr{D}^{(h)})^r$ of the division theorem proved in [1] for ideals in $\mathscr{D}^{(h)}$. Combined with Schreyer's method, this division theorem enables us to construct a filtered free resolution of any $(u,v)$-filtered graded $\mathscr{D}^{(h)}$-module $M'$ of finite type. We then obtain the result that any such $M'$ admits a $(u,v)$-filtered graded free resolution of length $\leqslant 2n+1$. We conclude this section by a description of the minimalization process which allows us to obtain a minimal free resolution from the Schreyer resolution. Therefore, an arbitrary minimal (resp. minimal $(u,v)$-filtered) resolution turns out to be of length $\leqslant 2n+1$.

If the module $M$ is defined by algebraic data, i.e., by operators with polynomial coefficients, then the computation of a minimal filtered resolution can be done in a completely algorithmic way by generalizing Mora's tangent cone algorithm to $\mathscr{D}^{(h)}$. This topic will be discussed elsewhere. An algebraic counterpart of minimal filtered resolution and its algorithm were considered in [7] without uniqueness result.

## 1. Minimal free resolutions of $\mathscr{D}^{(h)}$-modules

We denote by $\mathscr{D}$ the ring of germs of analytic linear differential operators (of finite order) at the origin of $\mathbb{C}^n$. Fixing the canonical coordinate system $x=(x_1,\ldots,x_n)$ of $\mathbb{C}^n$, we can write an element $P$ of $\mathscr{D}$ in a finite sum $P=\sum_{\beta\in\mathbb{N}^n} a_\beta(x)\partial^\beta$ with $a_\beta(x)$ belonging to $\mathbb{C}\{x\}$, the ring of convergent power series. Here, we use the notation $\partial^\beta=\partial_1^{\beta_1}\cdots\partial_n^{\beta_n}$ with $\partial_i=\partial/\partial x_i$ and $\beta=(\beta_1,\ldots,\beta_n)\in\mathbb{N}^n$, where $\mathbb{N}=\{0,1,2,\ldots\}$.

Following Assi et al. [1], we introduce the homogenized ring $\mathscr{D}^{(h)}$ of $\mathscr{D}$ as follows (in [1] it is denoted $\mathscr{D}[t]$): introducing a new variable $h$, we define the homogenization of $P \in \mathscr{D}$ above to be

$$P^{(h)} := \sum_{\beta} a_\beta(x)\partial^\beta h^{m-|\beta|} \quad \text{with}$$

$$m = \operatorname{ord} P := \max\{|\beta| = \beta_1 + \cdots + \beta_n | a_\beta(x) \neq 0\}.$$

We regard this operator as an element of the ring $\mathscr{D}^{(h)}$, which is the $\mathbb{C}$-algebra generated by $\mathbb{C}\{x\}$, $\partial_1, \ldots, \partial_n$, and $h$ with the commuting relations

$$ha = ah, \quad h\partial_i = \partial_i h, \quad \partial_i\partial_j = \partial_j\partial_i, \quad \partial_i a - a\partial_i = \frac{\partial a}{\partial x_i} h$$

for any $a \in \mathbb{C}\{x\}$ and $i, j \in \{1, \ldots, n\}$.

Note that $\mathscr{D}^{(h)}$ is isomorphic to the Rees ring of $\mathscr{D}$ with respect to the order filtration, i.e., the filtration by the degree in $\partial$ of operators, and is (left and right) Noetherian. The ring $\mathscr{D}^{(h)}$ is a non-commutative graded $\mathbb{C}$-algebra with the grading

$$\mathscr{D}^{(h)} = \bigoplus_{d \geqslant 0} (\mathscr{D}^{(h)})_d \quad \text{with} \quad (\mathscr{D}^{(h)})_d := \bigoplus_{|\beta|+k=d} \mathbb{C}\{x\}\partial^\beta h^k.$$

An element $P$ of $(\mathscr{D}^{(h)})_d$ is said to be homogeneous of degree $\deg P := d$. A left $\mathscr{D}^{(h)}$-module $M$ is a graded $\mathscr{D}^{(h)}$-module if $M$ is written as a direct sum $M = \oplus_{k\in\mathbb{Z}} M_k$ of $\mathbb{C}\{x\}$-modules such that $(\mathscr{D}^{(h)})_d M_k \subset M_{k+d}$ holds for any $k, d$. If $M$ is finitely generated, there exists an integer $k_0$ such that $M_k = 0$ for $k \leqslant k_0$. Let $M$ and $N$ be graded left $\mathscr{D}^{(h)}$-modules. Then a map $\psi : M \to N$ is a homomorphism of graded $\mathscr{D}^{(h)}$-modules (of degree 0) if $\psi$ is $\mathscr{D}^{(h)}$-linear and satisfies $\psi(M_i) \subset N_i$ for any $i \in \mathbb{Z}$.

The ring $\mathscr{D}^{(h)}$ has a unique maximal graded two-sided ideal

$$\mathscr{I}^{(h)} := (\mathbb{C}\{x\}x_1 + \cdots + \mathbb{C}\{x\}x_n) \oplus \bigoplus_{d\geqslant 1} (\mathscr{D}^{(h)})_d.$$

The quotient ring $\mathscr{D}^{(h)}/\mathscr{I}^{(h)}$ is isomorphic to $\mathbb{C}$.

**Lemma 1.1** (Nakayama's lemma for $\mathscr{D}^{(h)}$). *Let $M$ be a finitely generated graded left $\mathscr{D}^{(h)}$-module. Then $\mathscr{I}^{(h)}M = M$ implies $M = 0$.*

**Proof.** Suppose $M \neq \{0\}$ and let $\{f_1, \ldots, f_r\}$ be a minimal set of homogeneous generators of $M$. Since $f_r \in M = \mathscr{I}^{(h)}M$, there exist homogeneous $P_1, \ldots, P_r \in \mathscr{I}^{(h)}$ such that $f_r = P_1 f_1 + \cdots + P_r f_r$, which implies

$$(1 - P_r) f_r = P_1 f_1 + \cdots + P_{r-1} f_{r-1}.$$

By the homogeneity, $P_r$ belongs to $(\mathscr{D}^{(h)})_0 \cap \mathscr{I}^{(h)} = \mathbb{C}\{x\}x_1 + \cdots + \mathbb{C}\{x\}x_n$. Hence $1 - P_r$ is invertible in $\mathscr{D}^{(h)}$, which contradicts the minimality of the generating set. □

In view of this lemma, the classical theory of minimal free resolutions for modules over a local ring extends to graded left $\mathscr{D}^{(h)}$-modules.

By a *graded free $\mathscr{D}^{(h)}$-module $\mathscr{L}$* of rank $r$, we mean a graded $\mathscr{D}^{(h)}$-module

$$\mathscr{L} = (\mathscr{D}^{(h)})^r[\mathbf{n}] := \bigoplus_{d\in\mathbb{Z}} ((\mathscr{D}^{(h)})_{d-n_1} \oplus \cdots \oplus (\mathscr{D}^{(h)})_{d-n_r})$$

with an $\mathbf{n}=(n_1,\dots,n_r)\in\mathbb{Z}^r$, which we call the *shift vector* for $\mathscr{L}$. The unit vectors $(1,0,\dots,0),\dots,(0,\dots,0,1)\in\mathscr{L}$ are called the canonical generators of $\mathscr{L}$. An element $P=(P_1,\dots,P_r)$ of $(\mathscr{D}^{(h)})^r[\mathbf{n}]$ is said to be homogeneous of degree $\deg[\mathbf{n}](P)=d$ if each $P_i$ is homogeneous of degree $d-n_i$.

**Definition 1.2** (minimal free resolution). Let $M$ be a graded free left $\mathscr{D}^{(h)}$-module. An exact sequence

$$\cdots\to\mathscr{L}_2\xrightarrow{\psi_2}\mathscr{L}_1\xrightarrow{\psi_1}\mathscr{L}_0\xrightarrow{\psi_0}M\to 0$$

of graded left $\mathscr{D}^{(h)}$-modules, where $\mathscr{L}_i$ is a graded free $\mathscr{D}^{(h)}$-module of rank $r_i$, is called a *minimal free resolution* of $M$ if the maps in the induced complex

$$\cdots\to(\mathscr{D}^{(h)}/\mathscr{I}^{(h)})\otimes_{\mathscr{D}^{(h)}}\mathscr{L}_2\to(\mathscr{D}^{(h)}/\mathscr{I}^{(h)})\otimes_{\mathscr{D}^{(h)}}\mathscr{L}_1\to(\mathscr{D}^{(h)}/\mathscr{I}^{(h)})\otimes_{\mathscr{D}^{(h)}}\mathscr{L}_0$$

(of $\mathbb{C}$-vector spaces) are all zero; i.e., the entries of $\psi_i$ $(i\geqslant 1)$ regarded as a matrix belong to $\mathscr{I}^{(h)}$. By Nakayama's lemma, this is equivalent to the condition that the images by $\psi_i$ of the canonical generators of $\mathscr{L}_i$ is a minimal generating set of the kernel of $\psi_{i-1}$, or of $M$ if $i=0$, for each $i\geqslant 0$. We call the ranks $r_0,r_1,\dots$ the Betti numbers of $M$.

A minimal free resolution exists and is unique up to isomorphism. This follows from Nakayama's lemma as in the case with a local ring (see e.g., [3, Theorem 20.2]). In particular, the Betti numbers and the associated shift vectors (up to permutation of their components) are invariants of $M$.

**Example 1.3.** In two variables $(x_1,x_2)=(x,y)$, let $I^{(h)}$ be the left ideal of $\mathscr{D}^{(h)}$ generated by

$$x^3-y^2,\quad 2x\partial_x+3y\partial_y+6h,\quad 3x^2\partial_y+2y\partial_x$$

with $\partial_x=\partial/\partial x$, $\partial_y=\partial/\partial y$ and put $M:=\mathscr{D}^{(h)}/I^{(h)}$. Its dehomogenization $\mathscr{D}/(I^{(h)}|_{h=1})$ (see Section 3) is the local cohomology group supported by $x^3-y^2=0$. A minimal free resolution of $M$ is given by

$$0\to(\mathscr{D}^{(h)})^2[(1,1)]\xrightarrow{\psi_2}(\mathscr{D}^{(h)})^3[(0,1,1)]\xrightarrow{\psi_1}\mathscr{D}^{(h)}\xrightarrow{\psi_0}M\to 0$$

with $\psi_0$ being the canonical projection and

$$\psi_1=\begin{pmatrix}x^3-y^2\\ 2x\partial_x+3y\partial_y+6h\\ 3x^2\partial_y+2y\partial_x\end{pmatrix},\qquad \psi_2=\begin{pmatrix}-2\partial_x & x^2 & -y\\ 3\partial_y & y & -x\end{pmatrix}.$$

## 2. Minimal filtered free resolutions of $\mathscr{D}^{(h)}$-modules

We call $(u,v)=(u_1,\dots,u_n,v_1,\dots,v_n)\in\mathbb{Z}^{2n}$ a *weight vector* if $u_i+v_i\geqslant 0$ and $u_i\leqslant 0$ hold for $i=1,\dots,n$. For an element $P$ of $\mathscr{D}^{(h)}$ of the form

$$P=\sum_{k\geqslant 0,\alpha,\beta\in\mathbb{N}^n}a_{\alpha\beta k}x^\alpha\partial^\beta h^k\quad(a_{\alpha\beta k}\in\mathbb{C}),$$

where the sum is finite with respect to $k$ and $\beta$, we define the $(u,v)$-order of $P$ by

$$\mathrm{ord}_{(u,v)}(P) := \max\{\langle u,\alpha\rangle + \langle v,\beta\rangle \mid a_{\alpha\beta k} \neq 0 \text{ for some } k\}$$

with $\langle u,\alpha\rangle = u_1\alpha_1 + \cdots + u_n\alpha_n$. We also define the $(u,v)$-order of $P \in \mathscr{D}$ in the same way. (If $P = 0$, we define its $(u,v)$-order to be $-\infty$.) Then the $(u,v)$-filtrations of $\mathscr{D}$ and $\mathscr{D}^{(h)}$ are defined by

$$F_k^{(u,v)}(\mathscr{D}) := \{P \in \mathscr{D} \mid \mathrm{ord}_{(u,v)}(P) \leqslant k\},$$

$$F_k^{(u,v)}(\mathscr{D}^{(h)}) := \{P \in \mathscr{D}^{(h)} \mid \mathrm{ord}_{(u,v)}(P) \leqslant k\}$$

for $k \in \mathbb{Z}$.

For a $\mathbb{C}$-subspace $I$ of $\mathscr{D}$ or of $\mathscr{D}^{(h)}$, put $F_k(I) := F_k^{(u,v)}(\mathscr{D}) \cap I$ or $F_k^{(u,v)}(\mathscr{D}^{(h)}) \cap I$, respectively, and define

$$\mathrm{gr}_k^{(u,v)}(I) := F_k^{(u,v)}(I)/F_{k-1}^{(u,v)}(I).$$

Then the graded rings with respect to the $(u,v)$-filtration are defined by

$$\mathrm{gr}^{(u,v)}(\mathscr{D}) := \bigoplus_{k\in\mathbb{Z}} \mathrm{gr}_k^{(u,v)}(\mathscr{D}), \qquad \mathrm{gr}^{(u,v)}(\mathscr{D}^{(h)}) := \bigoplus_{k\in\mathbb{Z}} \mathrm{gr}_k^{(u,v)}(\mathscr{D}^{(h)}).$$

For an element $P$ of $F_k^{(u,v)}(\mathscr{D})$ or of $F_k^{(u,v)}(\mathscr{D}^{(h)})$, we denote by $\sigma^{(u,v)}(P) = \sigma_k^{(u,v)}(P)$ the residue class of $P$ in $\mathrm{gr}_k^{(u,v)}(\mathscr{D})$ or in $\mathrm{gr}_k^{(u,v)}(\mathscr{D}^{(h)})$.

The ring $\mathrm{gr}^{(u,v)}(\mathscr{D}^{(h)})$ is Noetherian and has a structure of bigraded $\mathbb{C}$-algebra

$$\mathrm{gr}^{(u,v)}(\mathscr{D}^{(h)}) = \bigoplus_{d\geqslant 0}\bigoplus_{k\in\mathbb{Z}} \mathrm{gr}_k^{(u,v)}((\mathscr{D}^{(h)})_d).$$

A non-zero element of $\mathrm{gr}_k^{(u,v)}((\mathscr{D}^{(h)})_d)$ is called bihomogeneous of bidegree $(d,k)$. In general, if $I$ is a graded left ideal of $\mathscr{D}^{(h)}$, then $\mathrm{gr}^{(u,v)}(I) := \bigoplus_{k\in\mathbb{Z}} \mathrm{gr}_k^{(u,v)}(I)$ is a bigraded left ideal of $\mathrm{gr}^{(u,v)}(\mathscr{D}^{(h)})$ with respect to the bigraded structure. In particular, $\mathrm{gr}^{(u,v)}(\mathscr{I}^{(h)})$ is the unique maximal bigraded (two-sided) ideal of $\mathrm{gr}^{(u,v)}(\mathscr{D}^{(h)})$. We can define the notion of a minimal free resolution of a bigraded $\mathrm{gr}^{(u,v)}(\mathscr{D}^{(h)})$-module by virtue of the following

**Lemma 2.1** (Nakayama's lemma for $\mathrm{gr}^{(u,v)}(\mathscr{D}^{(h)})$). *Let $M'$ be a finitely generated bigraded left $\mathrm{gr}^{(u,v)}(\mathscr{D}^{(h)})$-module. Then $\mathrm{gr}^{(u,v)}(\mathscr{I}^{(h)})M' = M'$ implies $M' = 0$.*

**Proof.** Let $\{f_1,\dots,f_r\}$ be a minimal generating set of $M'$ consisting of bihomogeneous elements. Then there exist bihomogeneous elements $P_1,\dots,P_r$ of $\mathrm{gr}^{(u,v)}(\mathscr{I}^{(h)})$ such that $f_r = P_1 f_1 + \cdots + P_r f_r$, which implies

$$(1 - P_r)f_r = P_1 f_1 + \cdots + P_{r-1} f_{r-1}.$$

By the bihomogeneity, $P_r$ belongs to

$$\mathrm{gr}_0^{(u,v)}(\mathscr{I}^{(h)} \cap (\mathscr{D}^{(h)})_0) = \mathrm{gr}_0^{(u,v)}(\mathbb{C}\{x\}x_1 + \cdots + \mathbb{C}\{x\}x_n) = \sum_{i=1}^{j} \mathbb{C}\{x_1,\dots,x_j\}x_i$$

if e.g., $u_i = 0$ for $1 \leqslant i \leqslant j$ and $u_i < 0$ for $j < i \leqslant n$. Hence $1 - P_r$ is invertible. This contradicts the minimality. $\square$

For a graded left $\mathscr{D}^{(h)}$-module $M = \oplus_{d\in\mathbb{Z}} M_d$, a family $\{F_k(M)\}_{k\in\mathbb{Z}}$ of $\mathbb{C}\{x\}$-submodules of $M$ satisfying

$$F_k(M) \subset F_{k+1}(M), \quad \bigcup_{k\in\mathbb{Z}} F_k(M) = M, \quad F_l^{(u,v)}(\mathscr{D}^{(h)})F_k(M) \subset F_{k+l}(M),$$

$$F_k(M) = \bigoplus_{d\in\mathbb{Z}} F_k(M_d) \quad \text{with} \quad F_k(M_d) := F_k(M) \cap M_d \quad (k,l,d \in \mathbb{Z})$$

is called a (graded) $(u,v)$-*filtration* of $M$. The graded module of $M$ associated with this filtration is a bigraded left $\mathrm{gr}^{(u,v)}(\mathscr{D}^{(h)})$-module defined by

$$\mathrm{gr}(M) := \bigoplus_{k\in\mathbb{Z}} \mathrm{gr}_k(M) = \bigoplus_{d\geqslant 0}\bigoplus_{k\in\mathbb{Z}} \mathrm{gr}_k(M_d),$$

$$\mathrm{gr}_k(M) := F_k(M)/F_{k-1}(M), \quad \mathrm{gr}_k(M_d) := F_k(M_d)/F_{k-1}(M_d).$$

A non-zero element of $\mathrm{gr}_k(M_d)$ is said to be bihomogeneous of bidegree $(d,k)$. For an element $f$ of $M$, we put $\mathrm{ord}_{(u,v)} f := \inf\{k \in \mathbb{Z} \mid f \in F_k(M)\}$.

A $(u,v)$-filtration $\{F_k(M)\}_{k\in\mathbb{Z}}$ on a graded $\mathscr{D}^{(h)}$-module $M$ is said to be *good* if there exist homogeneous elements $f_1,\dots,f_r$ of $M$ and integers $m_1,\dots,m_r$ such that

$$F_k(M) = F_{k-m_1}^{(u,v)}(\mathscr{D}^{(h)})f_1 + \cdots + F_{k-m_r}^{(u,v)}(\mathscr{D}^{(h)})f_r \quad (\forall k \in \mathbb{Z}).$$

Then $\mathrm{gr}(M)$ is finitely generated. A (good) filtration of a left $\mathscr{D}$-module and the associated graded module are defined in the same way, but without assuming any homogeneity for $M$ and $f_i$.

By assigning $\mathbf{m} = (m_1,\dots,m_r) \in \mathbb{Z}^r$, which we call a *shift vector* for the $(u,v)$-filtration, we define good filtrations on $\mathscr{D}^r$ and on $(\mathscr{D}^{(h)})^r$ by

$$F_k^{(u,v)}[\mathbf{m}](\mathscr{D}^r) := \{(P_1,\dots,P_r) \in \mathscr{D}^r \mid \mathrm{ord}_{(u,v)}(P_i) + m_i \leqslant k \ (i = 1,\dots,r)\},$$

$$F_k^{(u,v)}[\mathbf{m}]((\mathscr{D}^{(h)})^r) := \{(P_1,\dots,P_r) \in (\mathscr{D}^{(h)})^r \mid \mathrm{ord}_{(u,v)}(P_i) + m_i \leqslant k \ (i = 1,\dots,r)\},$$

respectively. For an element $P = (P_1,\dots,P_r)$ of $(\mathscr{D}^{(h)})^r$, we put

$$\mathrm{ord}_{(u,v)}[\mathbf{m}](P) := \min\{k \in \mathbb{Z} | P \in F_k^{(u,v)}[\mathbf{m}]((\mathscr{D}^{(h)})^r)\}$$

and denote by $\sigma^{(u,v)}[\mathbf{m}](P) = \sigma_k^{(u,v)}[\mathbf{m}](P)$ the residue class of $P \in F_k^{(u,v)}[\mathbf{m}]((\mathscr{D}^{(h)})^r)$ in $\mathrm{gr}_k^{(u,v)}[\mathbf{m}]((\mathscr{D}^{(h)})^r) := F_k^{(u,v)}[\mathbf{m}]((\mathscr{D}^{(h)})^r)/F_{k-1}^{(u,v)}[\mathbf{m}]((\mathscr{D}^{(h)})^r)$.

The free modules $\mathscr{D}^r$ and $(\mathscr{D}^{(h)})^r$ equipped with these filtrations are called $(u,v)$-filtered free modules. For a $(u,v)$-filtered free module $\mathscr{L} = (\mathscr{D}^{(h)})^r$ with the above filtration, we put

$$F_k(\mathscr{L}) := F_k^{(u,v)}[\mathbf{m}]((\mathscr{D}^{(h)})^r) \quad (k \in \mathbb{Z}).$$

The graded free module $\mathscr{L} = (\mathscr{D}^{(h)})^r[\mathbf{n}]$ equipped with this filtration is called a $(u,v)$-*filtered graded free module* and denoted by $\mathscr{L} = (\mathscr{D}^{(h)})^r[\mathbf{n}][\mathbf{m}]$ in order to explicitly refer to the shift vectors.

**Definition 2.2** (minimal filtered free resolution). Let $M$ be a graded left $\mathscr{D}^{(h)}$-module with a $(u,v)$-filtration $\{F_k(M)\}_{k\in\mathbb{Z}}$. Then a $(u,v)$-*filtered free resolution* of $M$ is an exact sequence

$$\cdots \to \mathscr{L}_2 \xrightarrow{\psi_2} \mathscr{L}_1 \xrightarrow{\psi_1} \mathscr{L}_0 \xrightarrow{\psi_0} M \to 0 \tag{2.1}$$

of graded left $\mathscr{D}^{(h)}$-modules with $(u,v)$-filtered graded free $\mathscr{D}^{(h)}$-modules $\mathscr{L}_i$ which induces an exact sequence

$$\cdots \to F_k(\mathscr{L}_2) \xrightarrow{\psi_2} F_k(\mathscr{L}_1) \xrightarrow{\psi_1} F_k(\mathscr{L}_0) \xrightarrow{\psi_0} F_k(M) \to 0$$

for any $k \in \mathbb{Z}$. Then (2.1) induces an exact sequence

$$\cdots \to \mathrm{gr}(\mathscr{L}_2) \xrightarrow{\bar{\psi}_2} \mathrm{gr}(\mathscr{L}_1) \xrightarrow{\bar{\psi}_1} \mathrm{gr}(\mathscr{L}_0) \xrightarrow{\bar{\psi}_0} \mathrm{gr}(M) \to 0 \tag{2.2}$$

of bigraded $\mathrm{gr}^{(u,v)}(\mathscr{D}^{(h)})$-modules. The filtered free resolution (2.1) is called a *minimal* $(u,v)$*-filtered free resolution* of $M$ if (2.2) is a minimal free resolution of $\mathrm{gr}(M)$.

This last condition means that, as a matrix, all entries of $\bar{\psi}_i$ $(i \geqslant 1)$ belong to $\mathrm{gr}^{(u,v)}(\mathscr{I}^{(h)})$ or equivalently that the image of the set of canonical generators of $\mathrm{gr}(\mathscr{L}_i)$ is a minimal set of generators of $\mathrm{Ker}\,\bar{\psi}_{i-1}$ for $i \geqslant 0$, and of $\mathrm{gr}(M)$ for $i = 0$. The existence of a minimal bigraded free resolution as (2.2) can be proved exactly in the same way as in the non-filtered case of Section 1, by using Lemma 2.1.

Note that a minimal $(\mathbf{0},\mathbf{1})$-filtered free resolution with $(\mathbf{0},\mathbf{1}) = (0,\ldots,0,1,\ldots,1)$ is simply a minimal free resolution. The main purpose of the rest of this section is to prove the existence and uniqueness of minimal filtered free resolutions. With this aim in mind, we first show fundamental properties of the $(u,v)$-filtration. The following lemma will play an essential role in our arguments.

**Lemma 2.3.** *Let $N$ be a graded submodule of a $(u,v)$-filtered graded free module $\mathscr{L} := (\mathscr{D}^{(h)})^r[\mathbf{n}][\mathbf{m}]$ equipped with the induced filtration $F_k(N) := N \cap F_k(\mathscr{L})$. Suppose that $P_1,\ldots,P_l$ are homogeneous elements of $N$ such that*

$$\mathrm{gr}(N) = \mathrm{gr}^{(u,v)}(\mathscr{D}^{(h)})\sigma^{(u,v)}[\mathbf{m}](P_1) + \cdots + \mathrm{gr}^{(u,v)}(\mathscr{D}^{(h)})\sigma^{(u,v)}[\mathbf{m}](P_l).$$

*Then, for any $P \in N$, there exist $Q_i \in \mathscr{D}^{(h)}$ such that*

$$P = \sum_{i=1}^{l} Q_i P_i, \quad \mathrm{ord}_{(u,v)}(Q_i) + \mathrm{ord}_{(u,v)}[\mathbf{m}](P_i) \leqslant \mathrm{ord}_{(u,v)}[\mathbf{m}](P) \quad (i = 1,\ldots,l).$$

**Proof.** Put $k := \mathrm{ord}_{(u,v)}[\mathbf{m}](P)$. We may assume that $P$ belongs to the degree $d$ component $N_d = \mathscr{L}_d \cap N$, which is a $\mathbb{C}\{x\}$-submodule of the free $\mathbb{C}\{x\}$-module $\mathscr{L}_d$ of finite rank. Let us consider the $\mathbb{C}\{x\}$-submodule

$$N'_d := \left\{ \sum_{i=1}^{l} A_i P_i \,\middle|\, A_i \text{ is homogeneous of degree } d - \deg P_i \text{ and } \mathrm{ord}_{(u,v)}[\mathbf{m}](A_i P_i) \leqslant k \right\}$$

of $N_d$. Using the assumption we can deduce that

$$P \in \bigcap_{j \geqslant 0} (F_{k-j}(\mathscr{L}_d) + N'_d).$$

Put $I := F_{-1}^{(u,v)}(\mathbb{C}\{x\})$. Then $I$ is a proper ideal of the local ring $\mathbb{C}\{x\}$ and $IF_{-i}(\mathscr{L}_d) = F_{-i-1}(\mathscr{L}_d)$ holds for $i$ large enough. Hence the Krull intersection theorem implies

$$\bigcap_{j\geqslant 0}(F_{k-j}(\mathscr{L}_d) + N'_d) = N'_d.$$

This completes the proof. □

In particular, this lemma implies that the $(u,v)$-filtration is Noetherian, or satisfies the Artin–Rees property.

**Lemma 2.4.** *Let $M$ be a graded $\mathscr{D}^{(h)}$-module with a good $(u,v)$-filtration $\{F_k(M)\}_{k\in\mathbb{Z}}$. Then, for any graded submodule $N$ of $M$, the induced filtration $F_k(N) := F_k(M)\cap N$ is also good.*

**Proof.** We can take a graded free $\mathscr{D}^{(h)}$-module $\mathscr{L}$ and a filtered graded homomorphism $\psi : \mathscr{L} \to M$ such that $\psi(F_k(\mathscr{L})) = F_k(M)$. Put $N' := \psi^{-1}(N)$ and $F_k(N') := F_k(\mathscr{L})\cap N'$. We can choose homogeneous elements $P_1,\dots,P_{r'}$ of $N'$ whose residue classes generate $\mathrm{gr}(N')$. Let $\psi' : (\mathscr{D}^{(h)})^{r'} \to N'$ be the graded homomorphism (with respect to an appropriate shift vector for $\mathscr{L}' := (\mathscr{D}^{(h)})^{r'}$) satisfying $\psi'(e_i) = P_i$, where $e_1,\dots,e_{r'}$ are the canonical generators of $\mathscr{L}'$. Then with the $(u,v)$-filtration on $\mathscr{L}'$ defined by an appropriate shift vector, we have $\psi'(F_k(\mathscr{L}'))\subset F_k(N')$ and $\psi'$ induces a surjective map $\bar{\psi}' : \mathrm{gr}(\mathscr{L}') \to \mathrm{gr}(N')$. Applying Lemma 2.3 to $P_1,\dots,P_{r'}$, we get $\psi'(F_k(\mathscr{L}'))=F_k(N')$. This implies $\psi(\psi'(F_k(\mathscr{L}'))) = \psi(F_k(N')) = F_k(N)$. □

We might be able to follow the standard arguments on filtered modules, especially on Noetherian and Zariskian filtered modules as is presented, e.g., in [9], in order to prove the following key proposition; in fact, the $(u,v)$-filtration is not Zariskian, but is 'graded Zariskian'. However, we choose to be self-contained below since in the proof of [9, Proposition 1.1.3d, p. 55], it seems that $\phi(L_k)$ is assumed to be closed without proof.

**Proposition 2.5.** *Let $M_0, M_1, M_2$ be $(u,v)$-filtered graded $\mathscr{D}^{(h)}$-modules and assume that the filtration of $M_1$ is good. Consider a complex*

$$M_2 \xrightarrow{\psi_2} M_1 \xrightarrow{\psi_1} M_0, \tag{2.3}$$

*where $\psi_1$ and $\psi_2$ are filtered graded homomorphisms. Under these assumptions, if the induced complex*

$$\mathrm{gr}(M_2) \xrightarrow{\bar{\psi}_2} \mathrm{gr}(M_1) \xrightarrow{\bar{\psi}_1} \mathrm{gr}(M_0)$$

*is exact, then complex (2.3) is filtered exact; i.e., the sequence*

$$F_k(M_2) \xrightarrow{\psi_2} F_k(M_1) \xrightarrow{\psi_1} F_k(M_0)$$

*is exact for any $k\in\mathbb{Z}$.*

**Proof.** Let $N$ be the kernel of $\psi_1$ with the filtration $\{F_k(N)\}_{k\in\mathbb{Z}}$ induced by that on $M_1$. Since this filtration is good by Lemma 2.4, there is a filtered graded free module

$\mathscr{L}$ and a homomorphism $\varphi : \mathscr{L} \to N$ such that $\varphi(F_k(\mathscr{L})) = F_k(N)$ holds for any $k \in \mathbb{Z}$. Let $e_1, \ldots, e_r$ be the canonical generators of $\mathscr{L}$ with $\mathrm{ord}_{(u,v)} e_i = m_i$ and put $f_i := \varphi(e_i)$.

Since the image of $\psi_2$ is contained in $N$, we can regard $\psi_2$ as a homomorphism $\psi_2 : M_2 \to N$. Then by the exactness of the graded complex we have $\mathrm{Im}\,\bar{\psi}_2 = \mathrm{Ker}\,\bar{\psi}_1 \supset \mathrm{gr}(\mathrm{Ker}\,\psi_1)$, and consequently $\bar{\psi}_2 :\ \mathrm{gr}(M_2) \to \mathrm{gr}(N)$ is surjective. Hence there exists $g_i \in F_{m_i}(M_2)$ such that its residue class $\bar{g}_i$ in $\mathrm{gr}_{m_i}(M_2)$ satisfies $\bar{\psi}_2(\bar{g}_i) = \bar{f}_i$, and consequently $\psi_2(g_i) - f_i$ belongs to $F_{m_i-1}(N) = \varphi(F_{m_i-1}(\mathscr{L}))$. It follows that there exist homogeneous $P_{ij} \in F^{(u,v)}_{m_i-m_j-1}(\mathscr{D}^{(h)})$ such that

$$\psi_2(g_i) = f_i + \sum_{j=1}^{r} P_{ij} f_j \quad (i = 1, \ldots, r).$$

Let $\psi : \mathscr{L} \to M_2$ and $\psi_2' : \mathscr{L} \to \mathscr{L}$ be the homomorphisms such that $\psi(e_i) = g_i$ and $\psi_2'(e_i) = e_i + \sum_{j=1}^{r} P_{ij} e_j$, respectively. Then $\psi$ and $\psi_2'$ are filtered graded homomorphisms which make the diagram

$$\begin{tikzcd} \mathscr{L} \arrow[r, "\psi_2'"] \arrow[d, "\psi"'] & \mathscr{L} \arrow[d, "\varphi"] \\ M_2 \arrow[r, "\psi_2"] & N \end{tikzcd}$$

commutative. Since $\psi_2'$ induces the identity map $\mathrm{gr}(\mathscr{L}) \to \mathrm{gr}(\mathscr{L})$, we have $\psi_2'(F_k(\mathscr{L})) = F_k(\mathscr{L})$ in view of Lemma 2.3. Hence, we get

$$F_k(N) = \varphi(F_k(\mathscr{L})) = (\varphi \circ \psi_2')(F_k(\mathscr{L})) = (\psi_2 \circ \psi)(F_k(\mathscr{L})) \subset \psi_2(F_k(M_2))$$

which implies $\psi_2(F_k(M_2)) = F_k(N) = F_k(M_1) \cap \mathrm{Ker}\,\psi_1$. This completes the proof. $\square$

This proposition immediately implies the following criterion for filtered free resolution.

**Proposition 2.6.** *Let $M$ be a graded left $\mathscr{D}^{(h)}$-module with a good $(u,v)$-filtration $\{F_k(M)\}_{k\in\mathbb{Z}}$. Assume that*

$$\cdots \to \mathscr{L}_3 \xrightarrow{\psi_3} \mathscr{L}_2 \xrightarrow{\psi_2} \mathscr{L}_1 \xrightarrow{\psi_1} \mathscr{L}_0 \xrightarrow{\psi_0} M \to 0 \qquad (2.4)$$

*is a complex of $(u,v)$-filtered graded left $\mathscr{D}^{(h)}$-modules, i.e., satisfying $\psi_i \circ \psi_{i+1} = 0$ and $\psi_i(F_k(\mathscr{L}_i)) \subset F_k(\mathscr{L}_{i-1})$ (here we put $\mathscr{L}_{-1} = M$) for any $i \geqslant 0$ and $k \in \mathbb{Z}$, with filtered graded free modules $\mathscr{L}_0, \mathscr{L}_1, \ldots$ (of finite type). Then complex (2.4) is a filtered free resolution of $M$ if and only if the complex*

$$\cdots \to \mathrm{gr}(\mathscr{L}_3) \xrightarrow{\bar{\psi}_3} \mathrm{gr}(\mathscr{L}_2) \xrightarrow{\bar{\psi}_2} \mathrm{gr}(\mathscr{L}_1) \xrightarrow{\bar{\psi}_1} \mathrm{gr}(\mathscr{L}_0) \xrightarrow{\bar{\psi}_0} \mathrm{gr}(M) \to 0$$

*induced by (2.4) is exact.*

**Proposition 2.7** (lifting). *Let $M$ be a graded left $\mathscr{D}^{(h)}$-module with a good $(u,v)$-filtration $\{F_k(M)\}_{k\in\mathbb{Z}}$. Let $\mathscr{L}_0$ be a $(u,v)$-filtered graded free $\mathscr{D}^{(h)}$-module and $\psi_0$: $\mathscr{L}_0 \to M$ be a graded $\mathscr{D}^{(h)}$-homomorphism such that $\psi_0(F_k(\mathscr{L}_0)) \subset F_k(M)$ for any*

*$k \in \mathbb{Z}$. Suppose that there exist $(u,v)$-filtered graded free $\mathscr{D}^{(h)}$-modules $\mathscr{L}_i$ for $i \geqslant 1$ and a graded free resolution*

$$\cdots \to \mathrm{gr}(\mathscr{L}_3) \xrightarrow{\varphi_3} \mathrm{gr}(\mathscr{L}_2) \xrightarrow{\varphi_2} \mathrm{gr}(\mathscr{L}_1) \xrightarrow{\varphi_1} \mathrm{gr}(\mathscr{L}_0) \xrightarrow{\varphi_0} \mathrm{gr}(M) \to 0 \tag{2.5}$$

*of* $\mathrm{gr}(M)$, *where $\varphi_0$ coincides with the homomorphism induced by $\psi_0$. Then there exist filtered graded homomorphisms $\psi_i : \mathscr{L}_i \to \mathscr{L}_{i-1}$ for $i \geqslant 1$ which induce $\varphi_i$ and make the sequence*

$$\cdots \to \mathscr{L}_3 \xrightarrow{\psi_3} \mathscr{L}_2 \xrightarrow{\psi_2} \mathscr{L}_1 \xrightarrow{\psi_1} \mathscr{L}_0 \xrightarrow{\psi_0} M \to 0$$

*a filtered free resolution of $M$.*

**Proof.** First, we construct $\psi_1$. Applying Proposition 2.5 to the sequence $\mathscr{L}_0 \to M \to 0$, we know that $\psi_0(F_k(\mathscr{L}_0)) = F_k(M)$ holds for any $k \in \mathbb{Z}$. Let $e_1, \ldots, e_{r_1}$ be the canonical generators of $\mathscr{L}_1$ with $e_j \in F_{m_j}(\mathscr{L}_1) \setminus F_{m_j - 1}(\mathscr{L}_1)$. For each $j$, we can choose a homogeneous element $P_j$ of $F_{m_j}(\mathscr{L}_0)$ whose residue class $\bar{P}_j$ in $\mathrm{gr}(\mathscr{L}_0)$ coincides with the image $\varphi_1(\bar{e}_j)$ of the residue class $\bar{e}_j$ of $e_j$ in $\mathrm{gr}(\mathscr{L}_1)$. Then $\bar{P}_j$ belongs to $\mathrm{Ker}\,\varphi_0$ since $\varphi_0(\bar{P}_j) = \varphi_0(\varphi_1(\bar{e}_j)) = 0$. This implies

$$\psi_0(P_j) \in F_{m_j - 1}(M) = \psi_0(F_{m_j - 1}(\mathscr{L}_0)).$$

Hence there exists a homogeneous element $Q_j$ of $F_{m_j - 1}(\mathscr{L}_0)$ such that $\psi_0(P_j - Q_j) = 0$. Let $\psi_1 : \mathscr{L}_1 \to \mathscr{L}_0$ be the graded $\mathscr{D}^{(h)}$-homomorphism such that $\psi_1(e_j) = P_j - Q_j$ for $j = 1, \ldots, r_1$. Then $\psi_1$ is a filtered homomorphism with $\psi_0 \circ \psi_1 = 0$ which induces $\varphi_1$. In view of Proposition 2.5, the sequence

$$F_k(\mathscr{L}_1) \xrightarrow{\psi_1} F_k(\mathscr{L}_0) \xrightarrow{\psi_0} F_k(M) \to 0$$

is exact for any $k \in \mathbb{Z}$. In the same way, we can successively construct filtered graded homomorphisms $\psi_i : \mathscr{L}_i \to \mathscr{L}_{i-1}$ for $i \geqslant 2$ which induce $\varphi_i$ such that $\psi_i(F_k(\mathscr{L}_i)) = F_k(\mathscr{L}_{i-1}) \cap \mathrm{Ker}\,\psi_{i-1}$ for any $k$. This completes the proof. □

**Lemma 2.8.** *Let $\mathscr{L}$ and $\mathscr{L}'$ be $(u,v)$-filtered graded free $\mathscr{D}^{(h)}$-modules and $\psi : \mathscr{L} \to \mathscr{L}'$ be a filtered graded homomorphism. Let $\bar{\psi} : \mathrm{gr}(\mathscr{L}) \to \mathrm{gr}(\mathscr{L}')$ be the induced bigraded homomorphism. Then $\psi$ is a filtered isomorphism, i.e., an isomorphism satisfying $\psi(F_k(\mathscr{L})) = F_k(\mathscr{L}')$ for any $k \in \mathbb{Z}$, if and only if $\bar{\psi}$ induces an isomorphism*

$$\bar{\bar{\psi}} : \mathrm{gr}(\mathscr{L})/\mathrm{gr}(\mathscr{I}^{(h)})\mathrm{gr}(\mathscr{L}) \to \mathrm{gr}(\mathscr{L}')/\mathrm{gr}(\mathscr{I}^{(h)})\mathrm{gr}(\mathscr{L}')$$

*of $\mathbb{C}$-vector spaces.*

**Proof.** We have only to prove the 'if' part. Assume $\bar{\bar{\psi}}$ is an isomorphism. Then, we have

$$\mathrm{gr}(\mathscr{I}^{(h)})(\mathrm{gr}(\mathscr{L}')/\mathrm{Im}\,\bar{\psi}) = (\mathrm{gr}(\mathscr{I}^{(h)})\mathrm{gr}(\mathscr{L}') + \mathrm{Im}\,\bar{\psi})/\mathrm{Im}\,\bar{\psi} = \mathrm{gr}(\mathscr{L}')/\mathrm{Im}\,\bar{\psi}.$$

This implies $\mathrm{Im}\,\bar{\psi} = \mathrm{gr}(\mathscr{L}')$ in view of Lemma 2.1, and consequently that $\psi$ is filtered surjective in view of Proposition 2.5. Hence, we can define a filtered graded homomorphism $\psi' : \mathscr{L}' \to \mathscr{L}$ such that $\psi \circ \psi'$ is the identity on $\mathscr{L}'$. In particular, $\psi'$ is injective. Since $\bar{\bar{\psi}}$ is a $\mathbb{C}$-isomorphism, so is $\bar{\bar{\psi}}'$. By the same argument as above, we can prove that $\psi'$ is surjective. Thus $\psi$ is a filtered isomorphism. $\square$

**Theorem 2.9** (existence and uniqueness of minimal filtered resolutions). *Let $M$ be a graded left $\mathscr{D}^{(h)}$-module with a good $(u,v)$-filtration $\{F_k(M)\}_{k\in\mathbb{Z}}$. Then there exists a minimal filtered free resolution*

$$\cdots \to \mathscr{L}_3 \xrightarrow{\psi_3} \mathscr{L}_2 \xrightarrow{\psi_2} \mathscr{L}_1 \xrightarrow{\psi_1} \mathscr{L}_0 \xrightarrow{\psi_0} M \to 0$$

*of $M$. Moreover, a minimal filtered free resolution of $M$ is unique up to isomorphism; i.e., if*

$$\cdots \to \mathscr{L}'_3 \xrightarrow{\psi'_3} \mathscr{L}'_2 \xrightarrow{\psi'_2} \mathscr{L}'_1 \xrightarrow{\psi'_1} \mathscr{L}'_0 \xrightarrow{\psi'_0} M \to 0$$

*is another minimal $(u,v)$-filtered free resolution of $M$, then there exist filtered graded $\mathscr{D}^{(h)}$-isomorphisms $\theta_i : \mathscr{L}_i \to \mathscr{L}'_i$, i.e., a graded $\mathscr{D}^{(h)}$-isomorphism (of degree 0) satisfying $\theta_i(F_k(\mathscr{L}_i)) = F_k(\mathscr{L}'_i)$ for any $k \in \mathbb{Z}$, such that the diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & \mathscr{L}_3 \arrow[r, "\psi_3"] \arrow[d, "\theta_3"'] & \mathscr{L}_2 \arrow[r, "\psi_2"] \arrow[d, "\theta_2"'] & \mathscr{L}_1 \arrow[r, "\psi_1"] \arrow[d, "\theta_1"'] & \mathscr{L}_0 \arrow[r, "\psi_0"] \arrow[d, "\theta_0"'] & M \arrow[d, equal] \\
\cdots \arrow[r] & \mathscr{L}'_3 \arrow[r, "\psi'_3"] & \mathscr{L}'_2 \arrow[r, "\psi'_2"] & \mathscr{L}'_1 \arrow[r, "\psi'_1"] & \mathscr{L}'_0 \arrow[r, "\psi'_0"] & M
\end{tikzcd}$$

*is commutative.*

**Proof.** First, let us prove the existence. Since $\mathrm{gr}(M)$ is finitely generated, we can take homogeneous $f_1,\ldots,f_{r_0} \in M$ such that their residue classes minimally generate $\mathrm{gr}(M)$. Let $\mathscr{L}_0$ be the $(u,v)$-filtered graded free module of rank $r_0$ with appropriate shifts so that the homomorphism $\psi_0 : \mathscr{L}_0 \to M$ which sends the canonical generators of $\mathscr{L}_0$ to $f_1,\ldots,f_{r_0}$ is graded and filtered. Starting with the bigraded homomorphism $\varphi_0$ induced by $\psi_0$, we can construct a minimal bigraded free resolution

$$\cdots \to \mathrm{gr}(\mathscr{L}_3) \xrightarrow{\varphi_3} \mathrm{gr}(\mathscr{L}_2) \xrightarrow{\varphi_2} \mathrm{gr}(\mathscr{L}_1) \xrightarrow{\varphi_1} \mathrm{gr}(\mathscr{L}_0) \xrightarrow{\varphi_0} \mathrm{gr}(M) \to 0.$$

Then Proposition 2.7 assures the existence of a minimal $(u,v)$-filtered free resolution of $M$ which induces the resolution of $\mathrm{gr}(M)$ above.

Now let us prove the uniqueness. We construct a filtered graded isomorphism $\theta_i : \mathscr{L}_i \to \mathscr{L}'_i$ by induction on $i$. Assume that we have already constructed filtered isomorphisms $\theta_0,\ldots,\theta_i$. Then, we have a commutative diagram

$$\begin{tikzcd}
\mathscr{L}_{i+1} \arrow[r, "\psi_{i+1}"] & \mathscr{L}_i \arrow[r, "\psi_i"] \arrow[d, "\theta_i"'] & \mathscr{L}_{i-1} \arrow[d, "\theta_{i-1}"'] \\
\mathscr{L}'_{i+1} \arrow[r, "\psi'_{i+1}"] & \mathscr{L}'_i \arrow[r, "\psi'_i"] & \mathscr{L}'_{i-1}
\end{tikzcd}$$

Here, for $i < 0$, we put $\mathscr{L}_{-1} = M$, $\mathscr{L}_{-2} = 0$ with $\theta_{-1}$ being the identity map of $M$.

Let $e_1,\dots,e_{r_{i+1}}$ be the canonical generators of $\mathscr{L}_{i+1}$ with $\mathrm{ord}_{(u,v)}(e_j)=m_j$. Then we have

$$\theta_i(\psi_{i+1}(e_j))\in \mathrm{Ker}\,\psi'_i\cap F_{m_j}(\mathscr{L}'_i)=\psi'_{i+1}(F_{m_j}(\mathscr{L}'_{i+1})).$$

Hence there exists a homogeneous element $P_j$ of $F_{m_j}(\mathscr{L}'_{i+1})$ such that $\psi'_{i+1}(P_j)=\theta_i(\psi_{i+1}(e_j))$. Let $\theta_{i+1}:\mathscr{L}_{i+1}\to\mathscr{L}'_{i+1}$ be the filtered graded homomorphism defined by $\theta_{i+1}(e_j)=P_j$ for $j=1,\dots,r_{i+1}$. Then $\psi'_{i+1}\circ\theta_{i+1}=\theta_i\circ\psi_{i+1}$ holds. In the same way, we can define a filtered graded homomorphism $\theta'_{i+1}:\mathscr{L}'_{i+1}\to\mathscr{L}_{i+1}$ such that $\psi_{i+1}\circ\theta'_{i+1}=\theta_i^{-1}\circ\psi'_{i+1}$. Denoting by $1_{\mathscr{L}_{i+1}}$ the identity map of $\mathscr{L}_{i+1}$, we get

$$\psi_{i+1}\circ(1_{\mathscr{L}_{i+1}}-\theta'_{i+1}\circ\theta_{i+1})=\psi_{i+1}-\theta_i^{-1}\circ\theta_i\circ\psi_{i+1}=0.$$

Hence denoting by $\bar\theta_{i+1}$ and $\bar\theta'_{i+1}$ the induced homomorphisms between $\mathrm{gr}(\mathscr{L}_{i+1})$ and $\mathrm{gr}(\mathscr{L}'_{i+1})$, we have

$$\mathrm{Im}(1_{\mathrm{gr}(\mathscr{L}_{i+1})}-\bar\theta'_{i+1}\circ\bar\theta_{i+1})\subset\mathrm{Ker}\,\bar\psi_{i+1}=\mathrm{Im}\,\bar\psi_{i+2}\subset\mathrm{gr}(\mathscr{I}^{(h)})\mathrm{gr}(\mathscr{L}_{i+1}).$$

This implies that $\bar\theta'_{i+1}\circ\ \bar\theta_{i+1}$ induces the identity endomorphism of $\mathrm{gr}(\mathscr{L}_{i+1})/\mathrm{gr}(\mathscr{I}^{(h)})$ $\mathrm{gr}(\mathscr{L}_{i+1})$. In the same way, we can show that $\bar\theta_{i+1}\circ\bar\theta'_{i+1}$ induces the identity endomorphism of $\mathrm{gr}(\mathscr{L}'_{i+1})/\mathrm{gr}(\mathscr{I}^{(h)})\mathrm{gr}(\mathscr{L}'_{i+1})$. In particular, $\bar\theta_{i+1}$ induces an isomorphism

$$\bar{\bar\theta}_{i+1}:\ \mathrm{gr}(\mathscr{L}_{i+1})/\mathrm{gr}(\mathscr{I}^{(h)})\mathrm{gr}(\mathscr{L}_{i+1})\to\mathrm{gr}(\mathscr{L}'_{i+1})/\mathrm{gr}(\mathscr{I}^{(h)})\mathrm{gr}(\mathscr{L}'_{i+1}).$$

Using Lemma 2.8, we conclude that $\theta_{i+1}$ is a filtered isomorphism. This completes the proof. □

**Corollary 2.10.** *For a graded $\mathscr{D}^{(h)}$-module $M$ with a good $(u,v)$-filtration, the rank $r_i$, the shifts $\mathbf{n}_i$ and $\mathbf{m}_i$ attached to the i-th free module $\mathscr{L}_i$ in a minimal $(u,v)$-filtered resolution of $M$ are unique (up to permutation of the components of each shift).*

**Proof.** Consider a $(u,v)$-filtered graded free module $\mathscr{L}=(\mathscr{D}^{(h)})^r[\mathbf{n}][\mathbf{m}]$ with $\mathbf{m}=(m_1,\dots,m_r)$ and $\mathbf{n}=(n_1,\dots,n_r)$. The bigraded structure of $\mathrm{gr}(\mathscr{L})$ makes $\mathrm{gr}(\mathscr{L})/\mathrm{gr}(\mathscr{I}^{(h)})\mathrm{gr}(\mathscr{L})$ a bigraded $\mathbb{C}$-vector space with respect to the bigrading $\mathbb{C}=\oplus_{i,j}\mathbb{C}_{i,j}$ of $\mathbb{C}$ with $\mathbb{C}_{0,0}=\mathbb{C}$ and $(\mathbb{C})_{i,j}=0$ for $(i,j)\neq(0,0)$. With respect to this bigraded structure, we have

$$\dim_{\mathbb{C}}(\mathrm{gr}(\mathscr{L})/\mathrm{gr}(\mathscr{I}^{(h)})\mathrm{gr}(\mathscr{L}))_{i,j}=\sharp\{k\in\{1,\dots,r\}\mid n_k=i,\ m_k=j\}.$$

This implies that $r$, $\mathbf{n}$, $\mathbf{m}$ remain unchanged under a filtered graded isomorphism up to permutation of the components in view of Lemma 2.8. □

As an application, let us take a germ $f(x)$ of analytic function at $0\in\mathbb{C}^n$ and let $I$ be the left ideal generated by $t-f(x)$, $\partial_i+(\partial f/\partial x_i)\partial_t$ $(i=1,\dots,n)$ with $\partial_i=\partial/\partial x_i$ and $\partial_t=\partial/\partial t$ in the ring $\mathscr{D}^{(h)}$ for the $n+1$ variables $(t,x)$. Consider the graded $\mathscr{D}^{(h)}$-module $M:=\mathscr{D}^{(h)}/I=\mathscr{D}^{(h)}\delta(t-f)$, where $\delta(t-f)$ is the residue class of 1. Note that the dehomogenization $M|_{h=1}$ is the local cohomology group supported by $t-f=0$.

Put $(u,v)=(-1,0,\dots,0;1,0,\dots,0)$, which corresponds to the V-filtration with respect to $t=0$. Then $M$ has a good V-filtration $F_k(M):=F_k^{(u,v)}(\mathscr{D}^{(h)})\delta(t-f)$ for $k\in\mathbb{Z}$. This filtered module $M$ is uniquely determined up to isomorphism by the divisor defined by $f$. In fact, if $a(x)$ is a germ of analytic function with $a(0) \neq 0$, then $\delta(t-a(x)f(x))$ is transformed to $a(x)^{-1}\delta(t'-f(x'))$ by the coordinate transformation $t' = a(x)^{-1}t$, $x' = x$, which preserves the V-filtration. Hence, the ranks and the shift vectors (up to permutation) are invariants of the divisor defined by $f$. The following examples were computed by using software kan/sm1 [5].

**Example 2.11.** Using the variables $(t, x, y)$, put $f = x^3 - y^2$. Then a minimal $(-1,0,0;1,0,0)$-filtered free resolution of $M := \mathscr{D}^{(h)}\delta(t-f)$ is given by

$$0 \to \mathscr{D}^{(h)}[\mathbf{n}_3][\mathbf{m}_3] \xrightarrow{\psi_3} (\mathscr{D}^{(h)})^5[\mathbf{n}_2][\mathbf{m}_2] \xrightarrow{\psi_2} (\mathscr{D}^{(h)})^5[\mathbf{n}_1][\mathbf{m}_1] \xrightarrow{\psi_1} \mathscr{D}^{(h)}[\mathbf{n}_0][\mathbf{m}_0] \xrightarrow{\psi_0} M \to 0$$

with shift vectors

$$\mathbf{n}_0 = (0), \quad \mathbf{n}_1 = (1,0,1,1,1), \quad \mathbf{n}_2 = (1,1,2,1,1), \quad \mathbf{n}_3 = (2),$$

$$\mathbf{m}_0 = (0), \quad \mathbf{m}_1 = (1,0,1,0,0), \quad \mathbf{m}_2 = (0,0,1,1,1), \quad \mathbf{m}_3 = (1).$$

The homomorphisms are given by the following matrices ($\psi_0$ is the canonical projection):

$$\psi_1 = \begin{pmatrix} \underline{-2y\partial_t} + \partial_y \\ \underline{-x^3 + y^2} + t \\ \underline{3x^2\partial_t} + \partial_x \\ \underline{3x^2\partial_y + 2y\partial_x} \\ \underline{-6t\partial_t - 2x\partial_x - 3y\partial_y - 6h} \end{pmatrix},$$

$$\psi_2 = \begin{pmatrix} 0 & \underline{-2\partial_x} & \underline{2t} & \underline{y} & \underline{x^2} \\ \underline{3t} & \underline{-3\partial_y} & 0 & \underline{-x} & \underline{-y} \\ \underline{\partial_x} & 0 & \underline{-\partial_y} & \underline{\partial_t} & 0 \\ \underline{-3x^2} & 0 & \underline{-2y} & 1 & 0 \\ \underline{3y} & \underline{6\partial_t} & \underline{2x} & 0 & 1 \end{pmatrix},$$

$$\psi_3 = (\underline{-6y\partial_t} + 3\partial_y, \quad \underline{-6x^2\partial_t} - 2\partial_x, \quad \underline{-6x^3 + 6y^2} + 6t,$$
$$\underline{-6t\partial_t - 2x\partial_x - 3y\partial_y - 5h}, \quad \underline{-3x^2\partial_y - 2y\partial_x}).$$

The underlined parts constitute the induced homomorphisms $\bar{\psi}_i$. Since the $b$-function of $x^3 - y^2$ has $-1$ as the only integer root, we get a minimal free resolution of Example 1.3 by restricting the above resolution to $t=0$ by using ($(\mathbf{0},\mathbf{1})$-homogenized version of) Algorithms 5.4, 7.3 of [6].

**Example 2.12.** The ranks of minimal V-filtered resolutions of $\mathscr{D}^{(h)}\delta(t-f)$ for several $f$ of three variables $x, y, z$ are as follows:

| $f$ | $r_0$ | $r_1$ | $r_2$ | $r_3$ | $r_4$ | $r_5$ |
|---|---|---|---|---|---|---|
| $xyz$ | 1 | 7 | 12 | 7 | 1 | 0 |
| $x^3+y^3+z^3$ | 1 | 8 | 12 | 7 | 2 | 0 |
| $x^3+y^2z^2$ | 1 | 8 | 13 | 8 | 2 | 0 |

## 3. Minimal filtered free resolutions of $\mathscr{D}$-modules

The aim of this section is to study minimal filtered free resolutions of $\mathscr{D}$-modules. The main result is that for a $\mathscr{D}$-module with a good $((\mathbf{0},\mathbf{1}),(u,v))$-bifiltration, its 'minimal' bifiltered free resolution can be defined uniquely up to isomorphism, via homogenization. Note that the $(u,v)$-filtrations on $\mathscr{D}$-modules are more delicate than those on $\mathscr{D}^{(h)}$-modules so that the arguments in the preceding section do not work in general.

The notion of a good bifiltration was introduced, for example, in [8,4]. The results below should be more or less 'well known to specialists', but we did not find them explicitly stated in the literature.

### 3.1. Correspondence between $(\mathbf{0},\mathbf{1})$-filtered $\mathscr{D}$-modules and graded $\mathscr{D}^{(h)}$-modules

For an element $P=\sum_{\beta,k} a_{\beta k}(x)\partial^\beta h^k$ of $\mathscr{D}^{(h)}$, we define its dehomogenization to be the element $\rho(P) := P|_{h=1} = \sum_{\beta,k} a_{\beta k}(x)\partial^\beta$ of $\mathscr{D}$. This defines a surjective ring homomorphism

$$\rho : \mathscr{D}^{(h)} \ni P \mapsto P|_{h=1} \in \mathscr{D},$$

which gives $\mathscr{D}$ a structure of two-sided $\mathscr{D}^{(h)}$-module. Let $M' = \oplus_{d\in\mathbb{Z}} M'_d$ be a graded left $\mathscr{D}^{(h)}$-module of finite type. Then we define its dehomogenization to be $\rho(M') := \mathscr{D}\otimes_{\mathscr{D}^{(h)}} M'$ as a left $\mathscr{D}$-module. Moreover, $\rho(M')$ has a good $(\mathbf{0},\mathbf{1})$-filtration (with $(\mathbf{0},\mathbf{1}) = (0,\dots,0;1,\dots,1)\in\mathbb{Z}^{2n}$) defined by

$$F_k^{(\mathbf{0},\mathbf{1})}(\rho(M')) := 1\otimes M'_k := \{1\otimes f \in \mathscr{D}\otimes M'_k \mid f\in M'_k\} \quad (k\in\mathbb{Z}) \tag{3.1}$$

in view of the relation $P\otimes f = 1\otimes(P^{(h)}f)$ and $\deg P^{(h)} = \operatorname{ord} P$ for $P\in\mathscr{D}$. For a graded homomorphism $\psi : M' \to N'$ of graded left $\mathscr{D}^{(h)}$-modules, $1\otimes\psi : \rho(M') \to \rho(N')$ is a filtered $\mathscr{D}$-homomorphism with the filtrations defined above. If $M'$ and $N'$ are free modules, then $1\otimes\psi$ is obtained by the substitution $\psi|_{h=1}$ of the matrix $\psi$. This defines a functor $\rho$ from the category of graded $\mathscr{D}^{(h)}$-modules of finite type, to the category of $\mathscr{D}$-modules with good $(\mathbf{0},\mathbf{1})$-filtrations.

A converse functor is given by the construction of so-called Rees modules. With a commutative variable $T$, the Rees ring of $\mathscr{D}$ with respect to the $(\mathbf{0},\mathbf{1})$-filtration is defined to be the $\mathbb{C}$-algebra

$$R_{(\mathbf{0},\mathbf{1})}(\mathscr{D}) := \bigoplus_{k\geqslant 0} F_k^{(\mathbf{0},\mathbf{1})}(\mathscr{D})T^k \subset \mathscr{D}[T],$$

which is isomorphic to $\mathscr{D}^{(h)}$ by the correspondence $a_\beta(x)\partial^\beta h^k \leftrightarrow a_\beta(x)\partial^\beta T^{k+|\beta|}$. Hence we identify $R_{(\mathbf{0},\mathbf{1})}(\mathscr{D})$ with $\mathscr{D}^{(h)}$. For a left $\mathscr{D}$-module with a good $(\mathbf{0},\mathbf{1})$-filtration $F_k^{(\mathbf{0},\mathbf{1})}(M)$, its Rees module is defined by

$$R_{(\mathbf{0},\mathbf{1})}(M) := \bigoplus_{k\in\mathbb{Z}} F_k^{(\mathbf{0},\mathbf{1})}(M)T^k,$$

which is a left $\mathscr{D}^{(h)}$-module of finite type with the action

$$a(x)\partial^\beta h^k(f_jT^j) = a(x)\partial^\beta f_jT^{j+k+|\beta|} \quad (a(x)\in\mathbb{C}\{x\},\ f_j\in F_j^{(\mathbf{0},\mathbf{1})}(M)).$$

If $\varphi: M\to N$ is a $(\mathbf{0},\mathbf{1})$-filtered homomorphism, this naturally induces a graded homomorphism

$$\varphi^{(h)}: R_{(\mathbf{0},\mathbf{1})}(M)\to R_{(\mathbf{0},\mathbf{1})}(N),$$

which is thought of as the homogenization of $\varphi$ with respect to both filtrations.

**Lemma 3.1.** *Let $f$ be a homogeneous element of a left graded $\mathscr{D}^{(h)}$-module $M'$. Then $1\otimes f=0$ holds in $\mathscr{D}\otimes_{\mathscr{D}^{(h)}} M'$ if and only if $h^\nu f = 0$ for some $\nu\in\mathbb{N}$.*

**Proof.** Assume $1\otimes f=0$. Then there exist a finite number of $Q_j\in\mathscr{D}$ and homogeneous $R_j\in\mathscr{D}^{(h)}$ such that $R_jf=0$ in $M'$ and $\sum_j Q_jR_j=1$ in $\mathscr{D}$ as a right $\mathscr{D}^{(h)}$-module. The latter relation reads $\sum_j Q_j\rho(R_j)=1$. Homogenizing this relation, we get $\sum_j h^{\nu_j}Q_j^{(h)}R_j = h^\nu$ with some $\nu,\nu_j\in\mathbb{N}$, which implies $h^\nu f=\sum_j h^{\nu_j}Q_j^{(h)}R_jf=0$. Conversely, if $h^\nu f=0$, we have $0=1\otimes h^\nu f=\rho(h^\nu)\otimes f=1\otimes f$. $\square$

**Definition 3.2.** A left $\mathscr{D}^{(h)}$-module $M'$ is called *h-saturated* if $hf=0$ implies $f=0$ for any homogeneous element $f$ of $M'$.

It is easy to see that the Rees module of a $(\mathbf{0},\mathbf{1})$-filtered $\mathscr{D}$-module $M$ is $h$-saturated.

**Proposition 3.3.** *The functors $\rho$* and *$R_{(\mathbf{0},\mathbf{1})}$ give an equivalence between the category of h-saturated graded $\mathscr{D}^{(h)}$-modules of finite type, and the category of $\mathscr{D}$-modules with good $(\mathbf{0},\mathbf{1})$-filtrations. Moreover these functors are exact, with exactness meaning filtered exactness in the second category.*

**Proof.** We omit the details of the proof which consists in proving that we have, in the sense of natural transformations of functors, two natural isomorphisms:

$$\Psi: M'\to R_{(\mathbf{0},\mathbf{1})}(\rho(M')),\quad \Phi:\mathscr{D}\otimes_{\mathscr{D}^{(h)}} R_{(\mathbf{0},\mathbf{1})}(M)\to M.$$

The verifications are straightforward.

The second point follows directly from this equivalence of categories and the fact that a short exact sequence is filtered exact precisely if and only if its image by the functor $R_{(\mathbf{0},\mathbf{1})}$ is exact. $\square$

From the arguments above, we have

**Theorem 3.4.** *Let $M$ be a left $\mathscr{D}$-module with a good $(\mathbf{0},\mathbf{1})$-filtration $\{F_k(M)\}_{k\in\mathbb{Z}}$. Then there exists a $(\mathbf{0},\mathbf{1})$-filtered free resolution*

$$\cdots \to \mathscr{L}_3 \xrightarrow{\varphi_3} \mathscr{L}_2 \xrightarrow{\varphi_2} \mathscr{L}_1 \xrightarrow{\varphi_1} \mathscr{L}_0 \xrightarrow{\varphi_0} M \to 0 \tag{3.2}$$

*with $\mathscr{L}_i = \mathscr{D}^{r_i}[\mathbf{n}_i]$ being the free module equipped with the $(\mathbf{0},\mathbf{1})$-filtration $\{F_d^{(\mathbf{0},\mathbf{1})}[\mathbf{n}_i](\mathscr{L}_i)\}_{d\in\mathbb{Z}}$ with $\mathbf{n}_i \in \mathbb{Z}^{r_i}$ such that*

$$\cdots \to R_{(\mathbf{0},\mathbf{1})}(\mathscr{L}_3) \xrightarrow{\varphi_3^{(h)}} R_{(\mathbf{0},\mathbf{1})}(\mathscr{L}_2) \xrightarrow{\varphi_2^{(h)}} R_{(\mathbf{0},\mathbf{1})}(\mathscr{L}_1) \xrightarrow{\varphi_1^{(h)}} R_{(\mathbf{0},\mathbf{1})}(\mathscr{L}_0) \xrightarrow{\varphi_0^{(h)}} R_{(\mathbf{0},\mathbf{1})}(M) \to 0$$

*is a minimal free resolution of $R_{(\mathbf{0},\mathbf{1})}(M)$. Moreover, such a free resolution as (3.2) is unique up to $(\mathbf{0},\mathbf{1})$-filtered isomorphism. Note that $R_{(\mathbf{0},\mathbf{1})}(\mathscr{L}_i)$ is isomorphic to the graded free module $(\mathscr{D}^{(h)})^{r_i}[\mathbf{n}_i]$.*

### *3.2. Correspondence between bifiltered $\mathscr{D}$-modules and $(u,v)$-filtered graded $\mathscr{D}^{(h)}$-modules*

In order to define the bifiltration, we will assume here that $(u,v) \neq (\mathbf{0},\mathbf{1})$. A $((\mathbf{0},\mathbf{1}),(u,v))$-bifiltration of a $\mathscr{D}$-module $M$ is a double sequence $\{F_{d,k}(M)\}_{d,k\in\mathbb{Z}}$ of $\mathbb{C}\{x\}$-submodules of $M$ such that

$$F_{d,k}(M) \subset F_{d+1,k}(M) \cap F_{d,k+1}(M), \qquad \bigcup_{d,k\in\mathbb{Z}} F_{d,k}(M) = M,$$

$$(F_{d'}^{(\mathbf{0},\mathbf{1})}(\mathscr{D}) \cap F_{k'}^{(u,v)}(\mathscr{D}))F_{d,k}(M) \subset F_{d+d',k+k'}(M) \quad \text{for any } d,d',k,k' \in \mathbb{Z}.$$

**Definition 3.5.** Let $M$ be a left $\mathscr{D}$-module. Then a bifiltration $\{F_{d,k}(M)\}$ is called a *good* $((\mathbf{0},\mathbf{1}),(u,v))$-*bifiltration* if there exist $f_1,\ldots,f_l \in M$ and $n_i, m_i \in \mathbb{Z}$ such that

$$F_{d,k}(M) = (F_{d-n_1}^{(\mathbf{0},\mathbf{1})}(\mathscr{D}) \cap F_{k-m_1}^{(u,v)}(\mathscr{D}))f_1 + \cdots + (F_{d-n_l}^{(\mathbf{0},\mathbf{1})}(\mathscr{D}) \cap F_{k-m_l}^{(u,v)}(\mathscr{D}))f_l$$

holds for any $d,k \in \mathbb{Z}$.

Forgetting the $(u,v)$-filtration gives (functorially) a $(\mathbf{0},\mathbf{1})$-filtration on $M$, by setting $F_d^{(\mathbf{0},\mathbf{1})}(M) = \bigcup_{k\in\mathbb{Z}} F_{d,k}(M)$, and this functor sends good bifiltered modules to good filtered modules.

Now let $M'$ be a $h$-saturated graded left $\mathscr{D}^{(h)}$-module with a good $(u,v)$-filtration $F_k^{(u,v)}(M')$. We can find $f_j' \in M'$ homogeneous of degree $n_j$, and integers $m_j$ so that

$$F_k^{(u,v)}(M') = F_{k-m_1}^{(u,v)}(\mathscr{D}^{(h)})f_1' + \cdots + F_{k-m_r}^{(u,v)}(\mathscr{D}^{(h)})f_r'$$

holds for any $k \in \mathbb{Z}$. Then $\rho(M')$ has a good bifiltration defined by

$$F_{d,k}(\rho(M')) := 1 \otimes F_k^{(u,v)}(M_d') = \sum_{j=1}^{r} (F_{d-n_j}^{(\mathbf{0},\mathbf{1})}(\mathscr{D}) \cap F_{k-m_j}^{(u,v)}(\mathscr{D}))(1 \otimes f_j').$$

We remark that this is not in general the intersection of the $d$-th term of the $(\mathbf{0},\mathbf{1})$-filtration on $\rho(M')$, which we denoted by $1 \otimes M_d'$, and the $k$th term of the $(u,v)$-filtration

$$F_k^{(u,v)}(\rho(M')) := 1 \otimes F_k^{(u,v)}(M') := \{1 \otimes f \in \mathscr{D} \otimes_{\mathscr{D}^{(h)}} M' \mid f \in F_k^{(u,v)}(M')\}.$$

Conversely, if $M$ is a left $\mathscr{D}$-module with a good bifiltration $\{F_{d,k}(M)\}$, then its Rees module $R_{(\mathbf{0},\mathbf{1})}(M)$ with respect to its $(\mathbf{0},\mathbf{1})$-filtration has a good $(u,v)$-filtration

$$F_k^{(u,v)}(R_{(\mathbf{0},\mathbf{1})}(M)) := \bigoplus_{d\in\mathbb{Z}} F_{d,k}(M)T^d = \sum_{j=1}^{r} F_{k-m_j}^{(u,v)}(\mathscr{D}^{(h)})(f_j T^{n_j}),$$

where $f_j, m_j, n_j$ are as in Definition 3.5.

**Proposition 3.6.** *The functors $\rho$ and $R_{(\mathbf{0},\mathbf{1})}$ give an equivalence between the category of $h$-saturated graded $\mathscr{D}^{(h)}$-modules with good $(u,v)$-filtrations and the category of $\mathscr{D}$-modules with good $((\mathbf{0},\mathbf{1}),(u,v))$-bifiltrations. Furthermore these functors are exact in the sense of filtered and bifiltered categories respectively.*

**Proof.** By the preceding discussion, we clearly obtain two functors between the categories defined in this proposition, which coincide with the two functors previously considered in Proposition 3.3 through the functors consisting in forgetting the $(u,v)$-filtrations. Since, the latter two functors give an equivalence, we have only to verify that the two mappings $\Psi, \Phi$ in the proof of Proposition 3.3 leave the $(u,v)$-filtrations (resp. the bifiltrations) unchanged, which is left to the reader.

Again the second point is simply a consequence of the equivalence of categories and of the fact the image of a *bifiltered* exact sequence by the functor $R_{(\mathbf{0},\mathbf{1})}$ is precisely a $(u,v)$-filtered and graded exact sequence. □

In conclusion we obtain

**Theorem 3.7.** *Let $M$ be a left $\mathscr{D}$-module with a good $((\mathbf{0},\mathbf{1}),(u,v))$-bifiltration $\{F_{d,k}(M)\}$. Then there exists a bifiltered free resolution*

$$\cdots \to \mathscr{L}_3 \xrightarrow{\varphi_3} \mathscr{L}_2 \xrightarrow{\varphi_2} \mathscr{L}_1 \xrightarrow{\varphi_1} \mathscr{L}_0 \xrightarrow{\varphi_0} M \to 0 \tag{3.3}$$

*with $\mathscr{L}_i = \mathscr{D}^{r_i}[\mathbf{n}_i][\mathbf{m}_i]$ being the free module equipped with the bifiltration $\{F_k^{(\mathbf{0},\mathbf{1})}[\mathbf{n}_i](\mathscr{L}_i) \cap F_k^{(u,v)}[\mathbf{m}_i](\mathscr{L}_i)\}$ with $\mathbf{n}_i, \mathbf{m}_i \in \mathbb{Z}^{r_i}$ such that*

$$\cdots \to R_{(\mathbf{0},\mathbf{1})}(\mathscr{L}_3) \xrightarrow{\varphi_3^{(h)}} R_{(\mathbf{0},\mathbf{1})}(\mathscr{L}_2) \xrightarrow{\varphi_2^{(h)}} R_{(\mathbf{0},\mathbf{1})}(\mathscr{L}_1) \xrightarrow{\varphi_1^{(h)}} R_{(\mathbf{0},\mathbf{1})}(\mathscr{L}_0) \xrightarrow{\varphi_0^{(h)}} R_{(\mathbf{0},\mathbf{1})}(M) \to 0$$

*is a minimal $(u,v)$-filtered free resolution of the $(u,v)$-filtered module $R_{(\mathbf{0},\mathbf{1})}(M)$. Moreover, such a free resolution as (3.3) is unique up to bifiltered isomorphism. Note that $R_{(\mathbf{0},\mathbf{1})}(\mathscr{L}_i)$ is isomorphic to the $(u,v)$-filtered graded free module $(\mathscr{D}^{(h)})^{r_i}[\mathbf{n}_i][\mathbf{m}_i]$.*

## 4. Division theorem for submodules of $(\mathscr{D}^{(h)})^r$

The aim of this chapter is to prove a division theorem (Theorem 4.1) for modules, which is a generalization of the similar theorem for ideals proved in [1], and to deduce from it the existence of a minimal filtered resolution of length $\leqslant 2n+1$ by a Schreyer type argument exactly as in [6], or in [3] for the commutative case. Except for Lemma 4.2, for which the choice of adapted linear forms is more complicated than in [1] in order to be suitable for all the orderings $<_L$ used in the Schreyer's type argument, the proof of the division theorem follows the lines of the proof in [1] with suitable modifications (see in particular Lemma 4.3); we shall sketch it only briefly.

### 4.1. Notations and statement of the division theorem

We fix a weight vector $(u,v)\in\mathbb{Z}^{2n}$, shift vectors $\mathbf{m},\mathbf{n},\mathbf{p}\in\mathbb{Z}^r$ with $\mathbf{m}=(m_1,\ldots,m_r)$ etc., and shift exponents $\mathbf{v}_i\in\mathbb{N}^{2n}$. We take also a well ordering $<_1$ on $\mathbb{N}^{2n}$ which is compatible with addition, and an ordering $<_2$ on the indices $\{1,\ldots,r\}$. Let $L=L_{(u,v)}$ be the linear form $L(\alpha,\beta)=\langle u,\alpha\rangle+\langle v,\beta\rangle$ on $\mathbb{Q}^{2n}$ associated with $(u,v)$.

We are going to define an ordering $<_L$ on the set $\mathbb{N}^{2n+1}\times\{1,\ldots,r\}$ for monomials in $(\mathscr{D}^{(h)})^r$, which is adapted to the fact that $(\mathscr{D}^{(h)})^r$ is considered as $(u,v)$-filtered and graded with respective shift vectors $\mathbf{m}$ and $\mathbf{n}$. The other shifts $\mathbf{p},\mathbf{v}_i$ are present in order to get a family of orderings which is stable in the Schreyer's argument, as we shall see in Section 4.3.

We consider the mapping $[L_1,\ldots,L_5]$: $\mathbb{N}^{2n+1}\times\{1,\ldots,r\}\to\mathbb{N}\times\mathbb{N}\times\mathbb{N}\times\mathbb{N}^{2n}\times\mathbb{N}$, and the product ordering $(\prec_1,\ldots,\prec_5)=(<,<,<,(<_1)^{\mathrm{opp}},<_2)$ on the target (opp denotes the opposite ordering), with the following list of $L_p(\alpha,\beta,k,i)$:

$$[L_1,\ldots,L_5]=[|\beta|+k+n_i,\ L(\alpha,\beta)+m_i,\ |\beta|+p_i,\ (\alpha,\beta)+\mathbf{v}_i,\ i]$$

We define $<_L$ in a lexicographical fashion by
$(\alpha,\beta,k,i)<_L(\alpha',\beta',k',j)$ if and only if there exists $p$ with $1\leqslant p\leqslant 5$ such that $L_q(\alpha,\beta,k,i)=L_q(\alpha',\beta',k',j)$ for any $q<p$, and $L_p(\alpha,\beta,k,i)\prec_p L_p(\alpha',\beta',k',j)$.

For an operator $P=\sum a_{\alpha,\beta,k,i}x^\alpha\partial^\beta h^k e_i$ with $e_1,\ldots,e_r$ being the canonical generators of $(\mathscr{D}^{(h)})^r$, its Newton diagram is defined by $\mathscr{N}(P):=\{(\alpha,\beta,k,i)\,|\,a_{\alpha,\beta,k,i}\neq 0\}$. We also define

- The leading exponent: $\operatorname{lexp}(P)$ or $\operatorname{lexp}_{<_L}(P):=\max_{<_L}\mathscr{N}(P)$,
- The leading monomial: $\mathrm{LM}(P)$ or $\mathrm{LM}_{<_L}(P)=x^\alpha\xi^\beta h^k e_i$ with $(\alpha,\beta,k,i):=\operatorname{lexp}_{<_L}(P)$, where $\xi$ denotes the commutative variables corresponding to $\partial$.
- The leading term: $\mathrm{LT}(P)$ or $\mathrm{LT}_{<_L}(P):=a_{\operatorname{lexp}(P)}\mathrm{LM}(P)$; we often identify it with the corresponding 'monomial' in $\mathrm{gr}^{(u,v)}[\mathbf{m}]((\mathscr{D}^{(h)})^r)$.

There is an action $\mathbb{N}^{2n+1}\times(\mathbb{N}^{2n+1}\times\{1,\ldots,r\})\to\mathbb{N}^{2n+1}\times\{1,\ldots,r\}$ defined by

$$(\alpha',\beta',k')+(\alpha,\beta,k,i)=(\alpha+\alpha',\beta+\beta',k+k',i)$$

such that for non-zero $Q\in\mathscr{D}^{(h)}$ and $P\in(\mathscr{D}^{(h)})^r$ we have $\operatorname{lexp}_{<_L}(QP)=\operatorname{lexp}_{<_L}(Q)+\operatorname{lexp}_{<_L}(P)$, where $\operatorname{lexp}_{<_L}(Q)$ is given by the case $r=1$ without shifts of the ordering $<_L$.

We consider $l$ elements $P^{(1)},\ldots,P^{(l)}$ of $(\mathscr{D}^{(h)})^r[\mathbf{n}]$ homogeneous of degree $d_1,\ldots,d_l$ respectively. Let their leading exponents be

$$\operatorname{lexp}_{<_L}(P^{(j)})=(\alpha_j,\beta_j,k_j,i_j)\in\mathbb{N}^{2n}\times\mathbb{N}\times\{1,\ldots,r\}.$$

We define a partition of $\mathbb{N}^{2n} \times \mathbb{N} \times \{1,\dots,r\}$ by

$$\Delta_1 := \mathrm{lexp}_{<_L}(P^{(1)}) + \mathbb{N}^{2n} \times \mathbb{N},$$

$$\Delta_i := (\mathrm{lexp}_{<_L}(P^{(i)}) + \mathbb{N}^{2n} \times \mathbb{N}) \setminus \bigcup_{j=1}^{i-1} \Delta_j \quad (i = 2,\dots,l),$$

$$\bar{\Delta} := \mathbb{N}^{2n} \times \mathbb{N} \times \{1,\dots,r\} \setminus \bigcup_{j=1}^{l} \Delta_j.$$

The following division theorem is a generalization to modules of the division theorem proved for left ideals of $\mathscr{D}^{(h)}$ in [1].

**Theorem 4.1** (Division theorem). *For any $P \in (\mathscr{D}^{(h)})^r$, there exist $Q_1,\dots,Q_l \in \mathscr{D}^{(h)}$ and $R \in (\mathscr{D}^{(h)})^r$ such that*

(1) $P = \sum_{i=1}^{l} Q_i P^{(i)} + R$,
(2) *If $Q_i \neq 0$, then $\mathscr{N}(Q_i) + \mathrm{lexp}_{<_L}(P^{(i)}) \subset \Delta_i$ for all $i$,*
(3) *If $R \neq 0$, then $\mathscr{N}(R) \subset \bar{\Delta}$.*

### *4.2. Proof of the division theorem*

Since the result depends only on the leading exponents $\mathrm{lexp}_{<_L}(P^{(i)})$, not on the ordering $<_L$ itself, we may assume in view of the following lemma that $u_i < 0$, $u_i + v_i > 0$ $(i = 1,\dots,n)$, and $\sigma^{(u,v)}[\mathbf{m}](P^{(j)}) = LT_{<_L}(P^{(j)})$.

**Lemma 4.2.** *Let $P_1,\dots,P_s$ be $s$ elements of $(\mathscr{D}^{(h)})^r$ and let an ordering $<_L$ be as in Section 4.1. Then there is a linear form $L' = L_{(u',v')}$ with $u'_i < 0$ and $u'_i + v'_i > 0$, and a shift vector $\mathbf{m}'$ such that the ordering $<_{L'}$ associated with the shifts $\mathbf{m}', \mathbf{n}, \mathbf{p}, \mathbf{v}_i$ satisfies the following two conditions:*

(1) *Each $P_j$ has the same leading term for $<_L$ and $<_{L'}$: $LT_{<_L}(P_j) = LT_{<_{L'}}(P_j)$.*
(2) *The principal $L'$-symbols of $P_j$ are monomials: $\sigma^{(u',v')}[\mathbf{m}'](P_j) = LT_{<_{L'}}(P_j)$.*

**Proof.** It is enough to work with only one operator $P = P_1$, and to make $s$ successive slight changes of the linear form $L$. There is a finite minimal set of exponents $A_l = (\alpha_l, \beta_l, k_l, i_l) \in \mathscr{N}(P)$ such that

$$\mathscr{N}(P) \subset \bigcup_{l=1}^{v} (A_l + \mathbb{N}^n \times (-\mathbb{N}^n) \times (-\mathbb{N})).$$

Then for any ordering $<_{L'}$, the leading exponent $\mathrm{lexp}_{<_{L'}}(P)$ is one of the $A_l$ among those with maximal degree $|\beta_l| + k_l + n_{i_l}$. In particular, we have $\mathrm{lexp}_{<_L}(P) = A_{l_0}$ with some $l_0$ and

$$A_l <_L A_{l_0} \quad \text{for any } l \neq l_0.$$

For the sake of simplicity of the notation, we put $A_{l_0} = (\alpha, \beta, k, i)$.

By Bayer [2], there exists a linear form $\Lambda$ on $\mathbb{Q}^{2n}$ with positive coefficients which order all the distinct exponents $(\alpha_l, \beta_l) + \mathbf{v}_{i_l}$ in the same way as $<_1$, so that

$$\Lambda(\alpha, \beta) + \Lambda(\mathbf{v}_i) < \Lambda(\alpha_l, \beta_l) + \Lambda(\mathbf{v}_{i_l}) \quad \text{if} \quad (\alpha, \beta) + \mathbf{v}_i <_1 (\alpha_l, \beta_l) + \mathbf{v}_{i_l}.$$

Let $\sigma$ be the permutation of $\{1, \ldots, r\}$ such that $i <_2 j$ if and only if $\sigma(i) < \sigma(j)$. Consider the linear form $L'(\alpha, \beta) = \langle u, \alpha \rangle + \langle v + \varepsilon \mathbf{1}, \beta \rangle - \varepsilon^2 \Lambda(\alpha, \beta)$. Then, we can verify easily that $L'$ satisfies the conclusion of the lemma for any $\varepsilon > 0$ small enough if we take the shift

$$\mathbf{m}' := \mathbf{m} + \varepsilon \mathbf{p} - \varepsilon^2 (\Lambda(\mathbf{v}_1), \ldots, \Lambda(\mathbf{v}_r)) + \varepsilon^3 (\sigma(1), \ldots, \sigma(r)). \qquad \square$$

Let $x^{\alpha_j} \xi^{\beta_j} h^{k_j} e_{i_j}$ be the monomial $LT_{<_L}(P^{(j)})$ and $\delta_j := \mathrm{ord}_{(u,v)}[\mathbf{m}](P^{(j)}) = L(\alpha_j, \beta_j) + m_{i_j}$. We consider the subspace $E_d \subset (\mathscr{D}^{(h)})^r[\mathbf{n}]$ of homogeneous vectors $P = (P_1, \ldots, P_r)$ of degree $d$. If $P = \sum p_{\alpha,\beta,k,i} x^\alpha \partial^\beta h^k e_i$ is such a vector, we define for any real number $s > 0$ its *pseudo-norm* by

$$|P|_s := \sum_{\alpha,\beta,k,i} |p_{\alpha,\beta,k,i}| s^{-L(\alpha,\beta) - m_i}.$$

We obtain in this way a family of Banach spaces $E_{d,s} \subset E_d$ defined by

$$E_{d,s} := \{P \in E_d \colon |P|_s < +\infty\}$$

such that $s \leqslant s' \Rightarrow E_{d,s'} \subset E_{d,s}$. Furthermore, we have $E_d = \bigcup_{s>0} E_{d,s}$. Indeed, because of the condition $u_i < 0$, the pseudo-norm is finite for given $P$ and $s$ small enough.

The following lemma, which is a partial step in a homogeneous division process, is a direct adaptation of Lemma 10 of [1].

**Lemma 4.3.** *Let $\hat{P}$ be an L-homogeneous element of $E_d$ with* $\mathrm{ord}_{(u,v)}[\mathbf{m}](\hat{P}) = \delta$. *Then, there exist $\hat{Q}_j \in \mathscr{D}^{(h)}$ $(j=1, \ldots, l)$ and $\hat{R}, \hat{S} \in (\mathscr{D}^{(h)})^r$ with* $\deg(Q_j) = d - d_j$, $\deg[\mathbf{n}](\hat{R}) = \deg[\mathbf{n}](\hat{S}) = d$ *which satisfy $\hat{P} = \sum_{j=1}^{l} \hat{Q}_j P^{(j)} + \hat{R} + \hat{S}$ and the following conditions*:

(1) *$\hat{Q}_j$ is $(u,v)$-homogeneous with* $\mathrm{ord}_{(u,v)}(\hat{Q}_j) = \delta - \delta_j$.
(2) *$\hat{R}$ is $F^{(u,v)}[\mathbf{m}]$-homogeneous with* $\mathrm{ord}_{(u,v)}[\mathbf{m}](\hat{R}) = \delta$.
(3) $\mathrm{ord}_{(u,v)}[\mathbf{m}](\hat{S}) \leqslant \delta - 1/h$ *for some fixed integer $h > 0$ such that $L(\mathbb{Z}^{2n}) \subset \frac{1}{h}\mathbb{Z}$.*
(4) $\mathscr{N}(\hat{Q}_j) + \mathrm{lexp}_{<_L}(P^{(j)}) \subset \Delta_j$ *and* $\mathscr{N}(\hat{R}) \subset \bar{\Delta}$.

*Moreover, there exist $K > 0$ and $s_0 > 0$, depending only on $P^{(1)}, \ldots, P^{(l)}$ and $d$, so that we have $|\hat{Q}_j|_s \leqslant s^{\delta_j} |\hat{P}|_s$, $|\hat{R}|_s \leqslant |\hat{P}|_s$, $|\hat{S}|_s \leqslant K s^{1/h} |\hat{P}|_s$ for all $0 < s \leqslant s_0$.*

Let $E_{d,s,\delta}$, resp. $E_{d,s,\leqslant\delta}$, be the space of elements of $E_{d,s}$ of $F^{(u,v)}[\mathbf{m}]$-order $\delta$, resp. $\leqslant \delta$. Then Lemma 4.3 gives us continuous linear mappings

$$q_{j,\delta} : E_{d,s,\delta} \to E^1_{d,s,\delta-\delta_j}, \quad r_\delta \colon E_{d,s,\delta} \to E_{d,s,\delta}, \quad v_\delta \colon E_{d,s,\delta} \to E_{d,s,\leqslant\delta-1/h}$$

defined by

$$q_{j,\delta}(\hat{P}) = \hat{Q}^{(j)}, \quad r_\delta(\hat{P}) = \hat{R}, \quad v_\delta(\hat{P}) = \hat{S},$$

where $E^1_{d,s,\delta}$ denotes the scalar version (with no shift) of $E_{d,s,\delta}$. By using these mappings, the final step of the proof is again almost the same as in [1].

### 4.3. Filtered free resolutions by Schreyer's method

Let $N$ be a $(u,v)$-filtered graded submodule of $(\mathscr{D}^{(h)})^r[\mathbf{n}][\mathbf{m}]$, and let $L = L_{(u,v)}$.

**Definition 4.4.** A subset $G = \{P^{(1)}, \dots, P^{(r_1)}\}$ of $N \setminus \{0\}$ is called a *Gröbner basis* or a *standard basis* of $N$ with respect to the ordering $<_L$ if it generates $N$ and satisfies

$$\mathrm{lexp}_{<_L}(N) := \{\mathrm{lexp}_{<_L}(P) \mid P \in N \setminus \{0\}\} = \bigcup_{i=1}^{r_1} (\mathrm{lexp}_{<_L}(P^{(i)}) + \mathbb{N}^{2n+1}).$$

We recall in Proposition 4.5 below a well-known criterion for a subset $G$ of $N$ to be a Gröbner basis. First put $\mathrm{lexp}_{<_L}(P^{(i)}) = (\alpha_i, \beta_i, k_i, l_i)$ and

$$\Lambda := \{(i,j) \mid 1 \leqslant i < j \leqslant r_1 \text{ and } l_i = l_j\}.$$

Consider the *S-vectors* $S_{j,i}P^{(i)} - S_{i,j}P^{(j)}$ of $(\mathscr{D}^{(h)})^r$ defined for $(i,j) \in \Lambda$ such that $S_{j,i}$ and $S_{i,j}$ are monomial operators that are minimal in the product partial ordering on $\mathbb{N}^{2n+1}$ satisfying

$$\mathrm{lexp}_{<_L}(S_{j,i}P^{(i)} - S_{i,j}P^{(j)}) <_L \mathrm{lexp}_{<_L}(S_{j,i}P^{(i)}) = \mathrm{lexp}(S_{i,j}P^{(j)}).$$

Applying Theorem 4.1 to the S-vectors, we get

$$S_{j,i}P^{(i)} - S_{i,j}P^{(j)} = \sum_{k=1}^{r_1} U_{i,j,k}P^{(k)} + R(i,j)$$

with $U_{i,j,k} \in \mathscr{D}^{(h)}$ and the remainder $R(i,j) \in (\mathscr{D}^{(h)})^r$.

By exactly the same argument as the proof of Proposition 16 of [1], we get

**Proposition 4.5.** *The following two conditions are equivalent*:

(1) *$G$ is a Gröbner basis of the module generated by $G$.*
(2) *For any $(i,j) \in \Lambda$, the remainder $R(i,j) = 0$.*

In the Schreyer method the ordering $<'_L$ on $\mathbb{N}^{2n+1} \times \{1, \dots, r_1\}$ adapted to the module of relations among $P^{(1)}, \dots, P^{(r_1)}$ is defined as follows:

$$(\alpha, \beta, k, i) <'_L (\alpha', \beta', k', i') \quad \text{if and only if}$$

$$\begin{cases} (\alpha, \beta, k) + \mathrm{lexp}_{<_L}(P^{(i)}) <_L (\alpha', \beta', k') + \mathrm{lexp}_{<_L}(P^{(i')}) \\ \text{or} \\ (\alpha, \beta, k) + \mathrm{lexp}_{<_L}(P^{(i)}) = (\alpha', \beta', k') + \mathrm{lexp}_{<_L}(P^{(i')}) \quad \text{and} \quad i' < i. \end{cases}$$

This ordering is of the same type as the one described in 4.1 with the same $L$, and shifts $\mathbf{n}', \mathbf{m}', \mathbf{p}' \in \mathbb{Z}^{r_1}$ and $\mathbf{v}'_i \in \mathbb{N}^{2n}$ $(i = 1, \ldots, r_1)$ for $<'_L$ being given by

$$n'_i := |\beta_i| + k_i + n_{l_i}, \quad m'_i := L(\alpha_i, \beta_i) + m_{l_i}, \quad p'_i := |\beta_i| + p_{l_i},$$

$$\mathbf{v}'_i := (\alpha_i, \beta_i) + \mathbf{v}_{l_i}.$$

Then we obtain the following theorem exactly in the same way as Theorem 9.10 of [6]:

**Theorem 4.6.** *Suppose that* $\{P^{(1)}, \ldots, P^{(r_1)}\}$ *is a Gröbner basis of* $N$ *with respect to* $<_L$. *Let us define elements of* $(\mathscr{D}^{(h)})^{r_1}$ *by*

$$V_{i,j} := (0, \ldots, S_{j,i}, \ldots, -S_{i,j}, \ldots, 0) - (U_{i,j,1}, \ldots, U_{i,j,r_1}).$$

*Then* $\{V_{i,j} \mid (i,j) \in \Lambda\}$ *is a Gröbner basis of the module of relations*

$$Syz(P^{(1)}, \ldots, P^{(r_1)}) := \left\{ (V_1, \ldots, V_{r_1}) \mid \sum_{i=1}^{r_1} V_j P^{(j)} = 0 \right\}$$

*with respect to the ordering* $<'_L$.

Using this process repeatedly, we can construct a $(u,v)$-filtered graded free resolution

$$\cdots \xrightarrow{\psi_3} (\mathscr{D}^{(h)})^{r_2} \xrightarrow{\psi_2} (\mathscr{D}^{(h)})^{r_1} \xrightarrow{\psi_1} (\mathscr{D}^{(h)})^{r_0} \xrightarrow{\psi_0} (\mathscr{D}^{(h)})^{r_0}/N \to 0.$$

Finally, by ordering the elements of the Gröbner basis at each step appropriately, we can arrange for the leading exponents of the rows of $\psi_{i+1}$ to depend only on $2n+1-i$ variables among $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n, h)$ for each $i \leqslant 2n+1$. (See [3, Corollary 15.11].) Since the leading exponents of the rows of $\psi_{2n+2}$ are of the type $(0, \ldots, 0; i)$, the division theorem implies that Coker $\psi_{2n+2}$ is filtered free. Thus we obtain

**Theorem 4.7.** *Any well-*$(u,v)$*-filtered graded* $\mathscr{D}^{(h)}$*-module* $M$ *admits a filtered free resolution of length* $\leqslant 2n+1$, *which can be obtained by Schreyer's process described above.*

*4.4. Step by step minimalization*

Let us consider a $(u,v)$-filtered graded free resolution

$$\cdots \xrightarrow{\psi_3} (\mathscr{D}^{(h)})^{r_2} \xrightarrow{\psi_2} (\mathscr{D}^{(h)})^{r_1} \xrightarrow{\psi_1} (\mathscr{D}^{(h)})^{r_0} \xrightarrow{\psi_0} (\mathscr{D}^{(h)})^{r_0}/N \to 0,$$

which is not necessarily minimal. We denote by $A_i = (P_{k,l})$ the $r_i \times r_{i-1}$ matrix defining $\psi_i$ as the right multiplication by $A_i$ for $i \geqslant 1$.

Denote by $\mathbf{n}_i = (n_{i,1}, \ldots, n_{i,r_i})$ and $\mathbf{m}_i = (m_{i,1}, \ldots, m_{i,r_i})$ the shifts which are involved in the above resolution in order for $\psi_i$ to be homogeneous and $(u,v)$-filtered. Then we have

$$\deg(P_{k,l}) = n_{i,k} - n_{i-1,l}, \quad \mathrm{ord}_{(u,v)}(P_{k,l}) \leqslant m_{i,k} - m_{i-1,l}.$$

The above resolution may, or may not, be the one obtained by Schreyer's method. In any case, we want to show how to obtain from it a minimal resolution in a constructive way. Assume that the given resolution is not minimal. This implies that the symbol $\sigma^{(u,v)}_{m_{i,k}-m_{i-1,l}}(P_{k,l})$ of some element of $A_i$ is not in $\mathrm{gr}_{(u,v)}(\mathscr{I}^{(h)})$. This symbol is bihomogeneous of bidegree $(0,0)$, which implies that $n_{i,k}=n_{i-1,l}$, $m_{i,k}=m_{i-1,l}$, and that it is an invertible power series involving only the variables $x_p$ of weight $u_p=0$. Since $P_{k,l}$ is homogeneous, it is also a power series, which is necessarily invertible.

We may assume without loss of generality that $k=l=1$, and we denote by $e_{i,1},\dots,e_{i,r_i}$ the canonical generators of $(\mathscr{D}^{(h)})^{r_i}$. Then we have a commutative diagram of $(u,v)$-filtered graded modules

$$\begin{tikzcd}
\cdots \arrow[r, "\psi_{i+2}"] & (\mathscr{D}^{(h)})^{r_{i+1}} \arrow[r, "\psi_{i+1}"] \arrow[d, equal] & (\mathscr{D}^{(h)})^{r_i} \arrow[r, "\psi_i"] \arrow[d, "p_i"'] & (\mathscr{D}^{(h)})^{r_{i-1}} \arrow[r, "\psi_{i-1}"] \arrow[d, "p_{i-1}"'] & (\mathscr{D}^{(h)})^{r_{i-2}} \arrow[r, "\psi_{i-2}"] \arrow[d, equal] & \cdots \\
\cdots \arrow[r, "\psi_{i+2}"] & (\mathscr{D}^{(h)})^{r_{i+1}} \arrow[r, "\psi'_{i+1}"] & (\mathscr{D}^{(h)})^{r_i-1} \arrow[r, "\psi'_i"] & (\mathscr{D}^{(h)})^{r_{i-1}-1} \arrow[r, "\psi'_{i-1}"] & (\mathscr{D}^{(h)})^{r_{i-2}} \arrow[r, "\psi_{i-2}"] & \cdots
\end{tikzcd} \tag{4.1}$$

In this diagram, the shifts for $(\mathscr{D}^{(h)})^{r_j-1}$ are $(n_{j,2},\dots,n_{j,r_j})$ and $(m_{j,2},\dots,m_{j,r_j})$ for $j=i,i-1$, the homomorphism $p_i$ is the projection onto the last $r_i-1$ factors, and $p_{i-1}$ is given by the composite

$$(\mathscr{D}^{(h)})^{r_{i-1}} \to \frac{(\mathscr{D}^{(h)})^{r_{i-1}}}{\mathscr{D}^{(h)}\cdot\psi_i(e_{i,1})} = \frac{(\mathscr{D}^{(h)})^{r_{i-1}}}{\mathscr{D}^{(h)}\cdot(P_{1,1},\dots,P_{1,r_{i-1}})} \simeq (\mathscr{D}^{(h)})^{r_{i-1}-1}$$

of the canonical projection and the filtered isomorphism which associates $(Q_2,\dots,Q_{r_{i-1}}) \in (\mathscr{D}^{(h)})^{r_{i-1}-1}$ with the residue class of $(0,Q_2,\dots,Q_{r_{i-1}})$ in $(\mathscr{D}^{(h)})^{r_{i-1}}/\mathscr{D}^{(h)}\cdot\psi_i(e_{i,1})$.

Since all the vertical arrows of (4.1) are filtered surjective, we can regard them as a morphism between the two horizontal complexes in (4.1), the kernel of which is isomorphic, as a $(u,v)$-filtered and graded complex, to the acyclic one

$$\cdots \to 0 \to \mathscr{D}^{(h)}e_{i,1} \to \mathscr{D}^{(h)}\psi_i(e_{i,1}) \to 0 \to \cdots.$$

Therefore this morphism is a $(u,v)$-filtered quasi-isomorphism and the lower horizontal complex of (4.1) is again a $(u,v)$-filtered free graded resolution of the module $M=(\mathscr{D}^{(h)})^{r_0}/N$. The matrices $A'_{i+1}$, $A'_i$, $A'_{i-1}$ representing the homomorphisms $\psi'_{i+1},\psi'_i,\psi'_{i-1}$ respectively are then given as follows:

- $A'_{i+1}$ is obtained from $A_{i+1}$ by removing its first column.
- $A'_{i-1}$ is obtained from $A_{i-1}$ by removing its first row, or is the appropriate quotient if $i-1=0$.
- $A'_i$ is slightly more complicated: it is an $(r_i-1)\times(r_{i-1}-1)$ matrix whose $(j-1,k-1)$-component is $P'_{j,k}=P_{j,k}-P_{j,1}P_{1,1}^{-1}P_{1,k}$.

Repeating this process a finite number of times, we get a desired minimal resolution.

## Acknowledgements

We acknowledge the financial support by the University of Angers, France, and JSPS, Japan, which made this collaboration possible.

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